

VORONOI TESSELLATIONS FOR MATCHBOX MANIFOLDS

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ABSTRACT. Matchbox manifolds \mathfrak{M} are a special class of foliated spaces, which includes as special examples the weak solenoids, and tiling spaces associated to aperiodic tilings with finite local complexity. They have many properties analogous to those of a compact manifold, but the additional data inherently encoded in the pseudogroup dynamics of its foliation \mathcal{F} represent fundamental groupoid data. As such, they are a rich class of mathematical objects to study. The special cases of weak solenoids and tiling spaces have an additional structure, that of a transverse foliation, consisting of a continuous family of Cantor sets transverse to the foliated structure. The purpose of this paper is to show that this transverse structure can be defined on all equicontinuous matchbox manifolds, as well as on special foliated subsets. This follows from the construction of uniform Voronoi tessellations on a dense leaf, which is the main goal of this work. From this we define a foliated Delaunay triangulation of \mathfrak{M} , adapted to the dynamics of \mathcal{F} . The result is highly technical, but underlies the study of the basic topological structure of matchbox manifolds in general. Our methods are similar to some prior results in the literature [30, 34], though are unique in that we give the construction of the Voronoi tessellations for a complete Riemannian manifold L of arbitrary dimension, while the constructions in the literature apply only to the case where the manifold L is Euclidean space \mathbb{R}^n . Thus, these technical results have interest in their own right, and have various further applications.

Key words and phrases. Voronoi tessellations, Delaunay triangulations, almost periodic tilings, laminations, solenoids, matchbox manifolds.

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1. INTRODUCTION

A *continuum* is a compact, connected, and non-empty metrizable space. An n -dimensional *foliated space* \mathfrak{M} is a continuum which has a local product structure; that is, every point of \mathfrak{M} has an open neighborhood homeomorphic to an open subset of \mathbb{R}^n times a compact metric space (the local transverse model). The leaves of the foliation \mathcal{F} of \mathfrak{M} are the maximal connected components with respect to the fine topology on \mathfrak{M} induced by the plaques of the local product structure. Precise definitions are given in section 2.

A *matchbox manifold* is a foliated space \mathfrak{M} with local transverse models that are totally disconnected, and the leaves have a smooth structure. This terminology was introduced in [1, 3, 4], and is based on the intuition that a 1-dimensional matchbox manifold \mathfrak{M} has local coordinate charts U which are homeomorphic to a “box of matches”, and for $n > 1$ the “matches” are n -dimensional. The notion of a lamination in the literature is essentially the same as that of a matchbox manifold, assuming that the transversals to the lamination are totally disconnected. Definition 2.4 below of a matchbox manifold makes precise various technical conditions imposed.

The classical solenoids modeled on \mathbb{S}^1 are 1-dimensional matchbox manifolds, and generalized solenoids with base an n -dimensional manifold, as in [26, 35, 41], are examples of n -dimensional matchbox manifolds. The tiling space associated to an aperiodic tiling of \mathbb{R}^n with finite local complexity provides another large class of examples. See [14, Chapter 11] and [?, 11, 12, 19, 20, 29, 36] for a variety of further examples.

For the special cases where \mathfrak{M} is a weak solenoid or tiling space, there is a natural inverse limit structure for \mathfrak{M} over a compact base M . The base M is a manifold in the case that \mathfrak{M} is a weak solenoid, and M is a branched manifold in the case that \mathfrak{M} is a tiling space. In both cases, there is given a continuous family of local transversals to the foliation \mathcal{F} on \mathfrak{M} . The collection of these transversals define a *transverse Cantor foliation* \mathcal{H} of \mathfrak{M} , whose “leaves” are themselves Cantor sets. This concept is made precise in Section 5.

The question we address in this paper, is when does a matchbox manifold \mathfrak{M} admit such a transverse Cantor foliation \mathcal{H} ? This is true in the two cases considered above. Our first result answers this question in the case of equicontinuous foliated spaces.

THEOREM 1.1. *Let \mathfrak{M} be an equicontinuous matchbox manifold. Then there exists a transverse Cantor foliation \mathcal{H} on \mathfrak{M} such that the projection to the leaf space $\mathfrak{M} \rightarrow \mathfrak{M}/\mathcal{H} \cong M$ is a Cantor bundle map over a compact manifold M .*

This is a fundamental result required for the work [18].

A fundamental result for the study of flows in several of the works of Aarts, Fokkink and Oversteegen [25, Lemma 5.2]; also, see the prior works [1, 2, 3, 26], is the “Long Box Lemma”, which states that every connected orbit segment K in a 1-dimensional matchbox manifold is contained in a bi-foliated open neighborhood \mathfrak{N}_K which is the “long box” neighborhood of K . Our next result generalizes this lemma to n -dimensional matchbox manifolds, for $n \geq 1$, and has a variety of applications [19, 20].

THEOREM 1.2 (Big Box). *Let \mathfrak{M} be a matchbox manifold, $L_x \subset \mathfrak{M}$ the leaf through $x \in \mathfrak{M}$, and \tilde{L}_x the holonomy covering of L_x . Let $\tilde{K} \subset \tilde{L}_x$ be a connected compact subset such that the composition $\iota_x: \tilde{K} \subset \tilde{L}_x \rightarrow L_x \subset \mathfrak{M}$ is injective with image K . Then there exists a clopen transversal $V_x \subset \mathfrak{M}$ containing x , and a foliated inclusion $\mathfrak{N}_K = K \times V_x \subset \mathfrak{M}$ such that the images $\{\{y\} \times V_x \mid y \in K\}$ form a continuous family of Cantor transversals \mathcal{H} to $\mathcal{F}|_{\mathfrak{N}_K}$.*

Moreover, if there is given a transverse foliation \mathcal{H}' which is partially defined on \mathfrak{N}_K , then modulo regularity assumptions, the foliation \mathcal{H} can be chosen to extend \mathcal{H}' .

That is, the set $\mathcal{F}|\mathfrak{N}_K$ is a bi-foliated neighborhood of K , so is a “Big Box” neighborhood of K . Note that in the case of the Long Box Lemma for flows, it is assumed that K is an orbit segment, so it has no holonomy. The requirement that K contains no loops with holonomy is essential.

If we consider the smooth analog of Theorem 1.2, where K is a compact subset without holonomy of a leaf in a smooth foliation of a manifold M , then the assertion analogous to Theorem 1.2 is simply a version of the Reeb Stability Theorem, and the existence of a product structure follows almost trivially from the existence of a transverse exponential map to the leaves.

However, for a matchbox manifold, there is no transverse exponential map, though one can try to create one. Candel and Conlon [14, Theorem 11.4.4] show that the foliated space \mathfrak{M} admits a smooth embedding $f: \mathfrak{M} \rightarrow \mathbb{R}^N$ for $N \gg 0$ sufficiently large. If one can show the existence of a foliation \mathcal{H} defined on an open neighborhood of the image $f(\mathfrak{M}) \subset \mathbb{R}^N$ which is uniformly transverse to the images of the leaves of \mathfrak{M} , then the approach used for smooth foliations can be applied. In fact, this suffices to give an alternate proof of Theorem 1.2. However, the proof of the existence of such a transverse foliation satisfying the required continuity condition on all of \mathfrak{M} seems to require a very detailed analysis of the “local geometry” of the embedding of \mathfrak{M} into a Hilbert space.

The proof of Theorem 1.1 given in this work uses only “elementary methods”, but are quite technical and involved due to a number of factors. The idea of the proof is straightforward enough. For each foliated coordinate chart, $\overline{U}_i \subset \mathfrak{M}$, there is a natural “vertical” foliation whose leaves are the images of the vertical transversals defined by the coordinate charts. The problem is that on the overlap of two charts, these vertical foliations need not match up, as the requirement on a foliation chart for \mathcal{F} is that the horizontal plaques match up. The exception is when \mathfrak{M} is given with a fibration structure, then the coordinates can be chosen to be adapted to the fibration structure, and so the fibers of the bundle are compatible on overlaps.

For the general case, the idea is then to subdivide the horizontal plaques into small enough regions, and restrict the diameters of the transverse model space for the chart, so that the vertical leaves become sufficiently close on overlaps, so that they can be made compatible on overlaps. More precisely, one constructs a uniform triangulation of the leaves of \mathcal{F} on \mathfrak{M} so that the triangles have sufficiently small diameter and in “general position”, so that they are stable in transverse directions, for small perturbations. The vertical foliations are then perturbed, using barycentric coordinates based on each simplex in the triangulation, so that their definitions are coordinate free, hence are globally defined.

A uniform triangulation of the leaves, satisfying required stability conditions, is constructed as the Delaunay simplicial complex associated to a Voronoi tessellation of the leaves, where the underlying net has sufficiently small spacing and is chosen to satisfy estimates imposed by the requirements of general position for the associated triangulation. The proof that all this can be done is quite tedious, and uses only “elementary techniques”, along with effective estimates in each stage of the process.

One reason for the technicalities encountered, is that we give effective estimates for the construction of Delaunay triangulations in the case where the leaves of \mathcal{F} are general Riemannian manifolds, and are not assumed to be Euclidean as in [34, 30] for example. Another reason, and more unexpected, is that the construction of “stable” Delaunay triangulations, as required by the foliation product structure, is fundamentally more subtle in dimensions greater than two. This leads to the most technically detailed aspects of this work. The authors present this construction in full detail, as it does not seem to be dealt with in the literature, yet is the foundation for a variety of other results. We note that our methods are related to the methods used by T. Giordano, H. Matui, I. Putnam, and C. Skau in their study of affability for \mathbb{Z}^d -Cantor minimal systems [30, 34], which develop a theory of Delaunay triangulations for higher dimensional Euclidean spaces. The methods of this paper are more geometric and so apply more generally, and hence of independent interest.

The results of this paper first appeared as Appendices A and B of the preprint [17], submitted for publication in June 2010. At the editor’s suggestion, the Appendices were removed from that work, and the results they contain on constructing foliated Delaunay triangulations for equicontinuous matchbox manifolds have been extended to a broader context in this work. Thus, this work is an essential companion to the paper [18].

The structure of the paper is as follows. In Part I give basic concepts and dynamical properties of matchbox manifolds. The proofs of results in Part 1, Sections 2 through Section 6, are generally either omitted, or when necessary for later development, they are briefly outlined, as the full proofs can be found in the companion paper [18].

Section 2 gives definitions and sets notations. Then Section 3 introduces the holonomy pseudogroup, and gives some basic technical properties of holonomy maps. Section 4 recalls important classical definitions from topological dynamics, adapted to the case of matchbox manifolds, and gives several results concerning the dynamical properties of matchbox manifolds. The notion of a *transverse Cantor foliation* \mathcal{H} of \mathfrak{M} is defined in Section 5. Finally, we conclude Part I of the paper with a discussion of the *Reeb Stability Theorem* for matchbox manifolds in Section 6, which is a basic concept in all study of foliation dynamics.

In Part II, Sections 7 and 8 give the classical constructions of Voronoi and Delaunay triangulations in the context of Riemannian manifolds. In Section 9, we extend these concepts from a single leaf, to a “parametrized version” which applies uniformly to the leaves of a matchbox manifold \mathfrak{M} .

In Part III, Sections 10 and 11 consider Euclidean Voronoi and Delaunay triangulations and their manipulations that are required in the subsequent constructions in the Riemannian context.

In Part IV, Sections 12 and 13 are the most technically demanding sections. The adaptation of the Voronoi tessellations, from a flat Euclidean structure to general Riemannian manifolds, requires the introduction of many painstaking estimates.

Finally, in Part V, Sections 14 and 15 establishes the existence of the transverse Cantor foliations, completing the proof of Theorem 1.1. In Section 16 we show how these constructions can be used to prove Theorem 1.2 and thereby establish a version of local Reeb product structure for matchbox manifolds.

PART I - MATCHBOX MANIFOLD DYNAMICS

In this part, we discuss the basic concepts of foliated spaces. Further discussion with examples can be found in [14, Chapter 11], [38, Chapter 2] and the first two authors’ papers [16, 18]. We give here precise definitions and formulate some of their basic geometric and dynamical properties. When appropriate, the reader is referred to the paper [18] for proofs, as there is significant overlap between the material in Part I, Sections 2 to 6 below, and the development given there.

2. FOLIATED SPACES

DEFINITION 2.1. *A continuum \mathfrak{M} is a foliated space of dimension n if there exists a compact separable metric space \mathfrak{X} , and for each $x \in \mathfrak{M}$ there is a compact subset $\mathfrak{T}_x \subset \mathfrak{X}$, open subset $U_x \subset \mathfrak{M}$, and homeomorphism defined on its closure $\varphi_x: \overline{U}_x \rightarrow [-1, 1]^n \times \mathfrak{T}_x$ such that $\varphi_x(x) = (0, w_x)$ where $w_x \in \text{int}(\mathfrak{T}_x)$. The subspace \mathfrak{T}_x of \mathfrak{X} is called the local transverse model at x .*

Let $\pi_x: \overline{U}_x \rightarrow \mathfrak{T}_x$ denote the composition of φ_x with projection onto the second factor.

For $w \in \mathfrak{T}_x$ the set $\mathcal{P}_x(w) = \pi_x^{-1}(w) \subset \overline{U}_x$ is called a *plaque* for the coordinate chart φ_x . We adopt the notation, for $z \in \overline{U}_x$ that $\mathcal{P}_x(z) = \mathcal{P}_x(\pi_x(z))$, so that $z \in \mathcal{P}_x(z)$. Note that each plaque $\mathcal{P}_x(w)$ is given the topology so that the restriction $\varphi_x: \mathcal{P}_x(w) \rightarrow [-1, 1]^n \times \{w\}$ is a homeomorphism. Then $\text{int}(\mathcal{P}_x(w)) = \varphi_x^{-1}((-1, 1)^n \times \{w\})$.

Let $U_x = \text{int}(\overline{U}_x) = \varphi_x^{-1}((-1, 1)^n \times \text{int}(\mathfrak{T}_x))$. Note that if $z \in U_x \cap U_y$, then $\text{int}(\mathcal{P}_x(z)) \cap \text{int}(\mathcal{P}_y(z))$ is an open subset of both $\mathcal{P}_x(z)$ and $\mathcal{P}_y(z)$. The collection of sets

$$\mathcal{V} = \{\varphi_x^{-1}(V \times \{w\}) \mid x \in \mathfrak{M}, w \in \mathfrak{T}_x, V \subset (-1, 1)^n \text{ open}\}$$

forms the basis for the *fine topology* of \mathfrak{M} . The connected components of the fine topology are called leaves, and define the foliation \mathcal{F} of \mathfrak{M} . For $x \in \mathfrak{M}$, let $L_x \subset \mathfrak{M}$ denote the leaf of \mathcal{F} containing x .

Note that in Definition 2.1, the collection of transverse models $\{\mathfrak{T}_x \mid x \in \mathfrak{M}\}$ need not have union equal to \mathfrak{X} . This is similar to the situation for a smooth foliation of codimension q , where each foliation chart projects to an open subset of \mathbb{R}^q , but the collection of images need not cover \mathbb{R}^q .

DEFINITION 2.2. *A smooth foliated space is a foliated space \mathfrak{M} as above, such that there exists a choice of local charts $\varphi_x: \overline{U}_x \rightarrow [-1, 1]^n \times \mathfrak{T}_x$ such that for all $x, y \in \mathfrak{M}$ with $z \in U_x \cap U_y$, there exists an open set $z \in V_z \subset U_x \cap U_y$ such that $\mathcal{P}_x(z) \cap V_z$ and $\mathcal{P}_y(z) \cap V_z$ are connected open sets, and the composition*

$$\psi_{x,y;z} \equiv \varphi_y^{-1} \circ \varphi_x: \varphi_x(\mathcal{P}_x(z) \cap V_z) \rightarrow \varphi_y(\mathcal{P}_y(z) \cap V_z)$$

is a smooth map, where $\varphi_x(\mathcal{P}_x(z) \cap V_z) \subset \mathbb{R}^n \times \{w\} \cong \mathbb{R}^n$ and $\varphi_y(\mathcal{P}_y(z) \cap V_z) \subset \mathbb{R}^n \times \{w'\} \cong \mathbb{R}^n$. The leafwise transition maps $\psi_{x,y;z}$ are assumed to depend continuously on z in the C^∞ -topology.

A map $f: \mathfrak{M} \rightarrow \mathbb{R}$ is said to be *smooth* if for each flow box $\varphi_x: \overline{U}_x \rightarrow [-1, 1]^n \times \mathfrak{T}_x$ and $w \in \mathfrak{T}_x$ the composition $\vec{y} \mapsto f \circ \varphi_x^{-1}(\vec{y} \times w)$ is a smooth function of $\vec{y} \in (-1, 1)^n$, and depends continuously on w in the C^∞ -topology on maps of the plaque coordinates \vec{y} . As noted in [38] and [14, Chapter 11], this allows one to define smooth partitions of unity, vector bundles, and tensors for smooth foliated spaces. In particular, one can define leafwise Riemannian metrics. We recall a standard result, whose proof for foliated spaces can be found in [14, Theorem 11.4.3].

THEOREM 2.3. *Let \mathfrak{M} be a smooth foliated space. Then there exists a leafwise Riemannian metric for \mathcal{F} , such that for each $x \in \mathfrak{M}$, L_x inherits the structure of a complete Riemannian manifold with bounded geometry, and the Riemannian geometry depends continuously on x .* \square

Bounded geometry implies, for example, that for each $x \in \mathfrak{M}$, there is a leafwise exponential map $\exp_x^{\mathcal{F}}: T_x \mathcal{F} \rightarrow L_x$ which is a surjection, and the composition $\exp_x^{\mathcal{F}}: T_x \mathcal{F} \rightarrow L_x \subset \mathfrak{M}$ depends continuously on x in the compact-open topology on maps.

DEFINITION 2.4. *A matchbox manifold is a continuum with the structure of a smooth foliated space \mathfrak{M} , such that for each $x \in \mathfrak{M}$, the transverse model space $\mathcal{T}_x \subset \mathfrak{X}$ is totally disconnected.*

2.1. Metric properties. For the rest of this paper, all foliated spaces are assumed to be smooth with a given leafwise Riemannian metric. The study of the dynamics of a foliated space \mathfrak{M} requires generalizing various concepts for flows, and group actions more generally, about the orbits of points in \mathfrak{M} , to the properties of leaves L of a foliation \mathcal{F} . On a technical level, it is very useful in developing these generalizations to have a strong local convexity property for the leaves, generalizing the local convexity of the orbit of a flow.

Another nuance about the definition of foliated spaces, and matchbox manifolds in particular, is that for given $x \in \mathfrak{M}$, the neighborhood \overline{U}_x in Definition 2.1 need not be “local”. As the transversal model \mathfrak{T}_x need not be connected, the set \overline{U}_x need not be connected, and *a priori* its connected components need not be contained in a metric ball around x . The following technical procedures ensure that we can always choose local charts for \mathfrak{M} to have a uniform locality property, as well as other metric regularity properties.

Let $d_{\mathfrak{M}}: \mathfrak{M} \times \mathfrak{M} \rightarrow [0, \infty)$ denote the metric on \mathfrak{M} , and $d_{\mathfrak{X}}: \mathfrak{X} \times \mathfrak{X} \rightarrow [0, \infty)$ the metric on \mathfrak{X} .

For $x \in \mathfrak{M}$ and $\epsilon > 0$, let $D_{\mathfrak{M}}(x, \epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x, y) \leq \epsilon\}$ be the closed ϵ -ball about x in \mathfrak{M} , and $B_{\mathfrak{M}}(x, \epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x, y) < \epsilon\}$ the open ϵ -ball about x .

Similarly, for $w \in \mathfrak{X}$ and $\epsilon > 0$, let $D_{\mathfrak{X}}(w, \epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w, w') \leq \epsilon\}$ be the closed ϵ -ball about w in \mathfrak{X} , and $B_{\mathfrak{X}}(w, \epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w, w') < \epsilon\}$ the open ϵ -ball about w .

Each leaf $L \subset \mathfrak{M}$ has a complete path-length metric induced from the leafwise Riemannian metric. That is, for $x, y \in L$ define

$$d_{\mathcal{F}}(x, y) = \inf \left\{ \|\gamma\| \mid \gamma: [0, 1] \rightarrow L \text{ is piecewise } C^1, \gamma(0) = x, \gamma(1) = y, \gamma(t) \in L \quad \forall 0 \leq t \leq 1 \right\}$$

and where $\|\gamma\|$ denotes the path length of the piecewise C^1 -curve $\gamma(t)$. If $x, y \in \mathfrak{M}$ and are not on the same leaf, then set $d_{\mathcal{F}}(x, y) = \infty$.

For each $x \in \mathfrak{M}$ and $r > 0$, let $D_{\mathcal{F}}(x, r) = \{y \in L_x \mid d_{\mathcal{F}}(x, y) \leq r\}$. The Gauss Lemma implies that there exists $\lambda_x > 0$ such that $D_{\mathcal{F}}(x, \lambda_x)$ is a *strongly convex* subset for the metric $d_{\mathcal{F}}$. That is, for any pair of points $y, y' \in D_{\mathcal{F}}(x, \lambda_x)$ there is a unique shortest geodesic segment in L_x joining y and y' and it is contained in $D_{\mathcal{F}}(x, \lambda_x)$ (cf. [22, Chapter 3, Proposition 4.2]). Note then, that for all $0 < \lambda < \lambda_x$ the disk $D_{\mathcal{F}}(x, \lambda)$ is also strongly convex.

LEMMA 2.5. *There exists $\lambda_{\mathcal{F}} > 0$ such that for all $x \in \mathfrak{M}$, $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}})$ is strongly convex.*

Proof. \mathfrak{M} is compact and the leafwise metrics have uniformly bounded geometry. \square

If \mathcal{F} is defined by a flow without periodic points, so that every leaf is diffeomorphic to \mathbb{R} , then the entire leaf is strongly convex, so $\lambda_{\mathcal{F}} > 0$ can be chosen arbitrarily. For foliations with leaves of dimension $n > 1$, the constant $\lambda_{\mathcal{F}}$ must be less than the injectivity radius for each of the leaves.

2.2. Regular covers. We next define what will be called a “regular covering” of \mathfrak{M} by foliation charts. This will be a finite collection of foliation charts which are well-adapted to the metrics $d_{\mathfrak{M}}$ on \mathfrak{M} and $d_{\mathfrak{X}}$ on \mathfrak{X} , and for the leafwise metric $d_{\mathcal{F}}$.

LEMMA 2.6. *There exists $\epsilon_{\mathcal{F}} > 0$ such that for all $y \in \mathfrak{M}$, there exists a compact set $\widehat{U} \subset \mathfrak{M}$ such that $D_{\mathfrak{M}}(y, 3\epsilon_{\mathcal{F}}) \subset \text{int}(\widehat{U})$, and for each leaf L of \mathcal{F} , each connected component of $L \cap \widehat{U}$ is a strongly convex subset of L .* \square

Next, we choose a set of coordinate charts so that they have diameter at most $\epsilon_{\mathcal{F}}$. For each $x \in \mathfrak{M}$, we can assume (see [18, Section 3]) there are given $\epsilon''_x, \delta_x > 0$, $w_x \in \mathfrak{X}$ and a foliation chart $\varphi_x: \overline{U}_x \rightarrow [-1, 1]^n \times \mathfrak{T}_x$ such that $\overline{U}_x \subset B_{\mathfrak{M}}(x, \epsilon_{\mathcal{F}})$, and the plaques of φ_x are leafwise strongly convex subsets with diameter $2\delta_x \leq \lambda_{\mathcal{F}}/2$. The collection of open sets

$$\{U_x \equiv \text{int}(\overline{U}_x) = \varphi_x^{-1}((-1, 1)^n \times B_{\mathfrak{X}}(w_x, \epsilon''_x)) \mid x \in \mathfrak{M}\}$$

forms an open cover of the compact space \mathfrak{M} , so there exists a finite subcover “centered” at the points $\{x_1, \dots, x_{\nu}\}$ where $\varphi_{x_i}(x_i) = 0 \times w_{x_i}$ for $w_{x_i} \in \mathfrak{X}$. Set

$$(1) \quad \delta_{\mathcal{U}}^{\mathcal{F}} = \min\{\delta_{x_1}, \dots, \delta_{x_{\nu}}\}$$

Each open set \overline{U}_{x_j} can be covered by a finite collection of foliation charts with leafwise radius $\delta_{\mathcal{U}}^{\mathcal{F}}$.

We simplify notation as follows. For $1 \leq i \leq \nu$, set $\overline{U}_i = \overline{U}_{x_i}$, $U_i = U_{x_i}$, and $\epsilon_i = \epsilon''_{x_i}$. Let $\mathcal{U} = \{U_1, \dots, U_{\nu}\}$ denote the corresponding open covering of \mathfrak{M} . Then there are corresponding coordinate maps

$$\varphi_i = \varphi_{x_i}: \overline{U}_i \rightarrow [-1, 1]^n \times \mathfrak{T}_i \quad , \quad \pi_i = \pi_{x_i}: \overline{U}_i \rightarrow \mathfrak{T}_i \quad , \quad \lambda_i: \overline{U}_i \rightarrow [-1, 1]^n$$

For $z \in \overline{U}_i$, the plaque of the chart φ_i through z is denoted by $\mathcal{P}_i(z) = \mathcal{P}_i(\pi_i(z)) \subset \overline{U}_i$. Note that the restriction $\lambda_i: \mathcal{P}_i(z) \rightarrow [-1, 1]^n$ is a homeomorphism onto. Also, define sections

$$\tau_i: \mathfrak{T}_i \rightarrow \overline{U}_i \quad , \quad \text{defined by } \tau_i(\xi) = \varphi_i^{-1}(0 \times \xi) \quad , \quad \text{so that } \pi_i(\tau_i(\xi)) = \xi$$

Let \mathcal{T}_i denote the image of τ_i and let $\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_{\nu} \subset \mathfrak{M}$ denote their union.

Let $\mathfrak{T}_* = \mathfrak{T}_1 \cup \dots \cup \mathfrak{T}_{\nu} \subset \mathfrak{X}$; note that \mathfrak{T}_* is compact, and if each \mathfrak{T}_i is totally disconnected, then \mathfrak{T}_* will also be totally disconnected.

DEFINITION 2.7. *A regular covering of a smooth foliated space \mathfrak{M} is a covering by foliation charts satisfying locality – each $\overline{U}_i \subset B_{\mathfrak{M}}(x_i, \epsilon_{\mathcal{F}})$ – and local convexity.*

We assume in the following that such a covering \mathcal{U} of \mathfrak{M} has been chosen. If \mathcal{F} is a smooth foliation of a compact manifold M , and $\mathfrak{M} \subset M$ is a closed saturated set, then the restriction to \mathfrak{M} of a regular covering for \mathcal{F} on M (see [14, Chapter 2]) defines a regular covering of the foliated space \mathfrak{M} .

LEMMA 2.8. *Suppose that $z \in U_i \cap U_j$ then $\mathcal{P}_i(z) \cap \mathcal{P}_j(z)$ is a strongly convex subset of L_x .* \square

2.3. Foliated maps. A *leafwise path*, or more precisely an \mathcal{F} -*path*, is a continuous map $\gamma: [0, 1] \rightarrow \mathfrak{M}$ such that there is a leaf L of \mathcal{F} for which $\gamma(t) \in L$ for all $0 \leq t \leq 1$.

LEMMA 2.9. *Let \mathfrak{M} be a matchbox manifold. A continuous map $\gamma: [0, 1] \rightarrow \mathfrak{M}$ is a leafwise path.*

Proof. Let $a \leq c \leq b$ and chose a local chart $\varphi_i: U_i \rightarrow (-1, 1)^n \times \mathfrak{T}_i$ with $\gamma(c) \in U_i$. The image path $\pi_i(\gamma(t)) \in \mathfrak{T}_i$ must be constant for t near to c , as \mathfrak{T}_i is assumed to be totally disconnected. Thus, by standard arguments, $\gamma(t)$ lies in the leaf L_x of \mathcal{F} containing the initial point $x = \gamma(a)$. \square

COROLLARY 2.10. *Let \mathfrak{M} be a matchbox manifold, X a path connected topological space, and $h: X \rightarrow \mathfrak{M}$ a continuous map. Then there exists a leaf $L_h \subset \mathfrak{M}$ for which $h(X) \subset L_h$.*

Proof. Let $x \in X$ and L_h be the leaf of \mathcal{F} containing $h(x)$. Then apply Lemma 2.9. \square

COROLLARY 2.11. *Let \mathfrak{M} and \mathfrak{M}' be matchbox manifolds, and $h: \mathfrak{M}' \rightarrow \mathfrak{M}$ a continuous map. Then h maps the leaves of \mathcal{F}' to leaves of \mathcal{F} .*

Proof. The leaves of \mathcal{F}' are path-connected, so their images under h are contained in leaves of \mathcal{F} . \square

COROLLARY 2.12. *A homeomorphism $h: \mathfrak{M} \rightarrow \mathfrak{M}$ of a matchbox manifold is a foliated map.* \square

2.4. Local estimates. We next introduce a number of constants based on the above choices, which will be used throughout the paper when making metric estimates.

Let $\epsilon_{\mathcal{U}} > 0$ be a Lebesgue number for the covering \mathcal{U} . That is, given any $z \in \mathfrak{M}$ there exists some index $1 \leq i_z \leq \nu$ such that the open metric ball $B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$.

The local projections $\pi_i: \overline{U}_i \rightarrow \mathfrak{T}_i$ and sections $\tau_i: \mathfrak{T}_i \rightarrow \overline{U}_i$ are continuous maps of compact spaces, so admit uniform metric estimates as follows.

LEMMA 2.13. *There exists a continuous increasing function ρ_{π} (the modulus of continuity for the projections π_i) such that:*

$$(2) \quad \forall 1 \leq i \leq \nu \text{ and } x, y \in \overline{U}_i \quad , \quad d_{\mathfrak{M}}(x, y) < \rho_{\pi}(\epsilon) \implies d_{\mathfrak{X}}(\pi_i(x), \pi_i(y)) < \epsilon \text{ .}$$

Proof. Set $\rho_{\pi}(\epsilon) = \min \{ \epsilon, \min \{ d_{\mathfrak{M}}(x, y) \mid 1 \leq i \leq \nu, x, y \in \overline{U}_i, d_{\mathfrak{X}}(\pi_i(x), \pi_i(y)) \geq \epsilon \} \}$. \square

LEMMA 2.14. *There exists a continuous increasing function ρ_{τ} (the modulus of continuity for the sections τ_i) such that:*

$$(3) \quad \forall 1 \leq i \leq \nu \text{ and } w, w' \in \mathfrak{T}_i \quad , \quad d_{\mathfrak{X}}(w, w') < \rho_{\tau}(\epsilon) \implies d_{\mathfrak{M}}(\tau_i(w), \tau_i(w')) < \epsilon \text{ .}$$

Proof. Set $\rho_{\tau}(\epsilon) = \min \{ \epsilon, \min \{ d_{\mathfrak{X}}(w, w') \mid 1 \leq i \leq \nu, w, w' \in \mathfrak{T}_i, d_{\mathfrak{M}}(\tau_i(w), \tau_i(w')) \geq \epsilon \} \}$. \square

Finally, we introduce two additional constants, derived from the Lebesgue number $\epsilon_{\mathcal{U}}$ chosen above.

The first is derived from a “converse” to the modulus function ρ_{π} . Set:

$$(4) \quad \epsilon_{\mathcal{U}}^{\mathcal{T}} = \max \{ \epsilon \mid \forall 1 \leq i \leq \nu, \forall x \in \overline{U}_i, D_{\mathfrak{M}}(x, \epsilon_{\mathcal{U}}/2) \subset \overline{U}_i, D_{\mathfrak{X}}(\pi_i(x), \epsilon) \subset \pi_i(D_{\mathfrak{M}}(x, \epsilon_{\mathcal{U}}/2)) \} \text{ .}$$

Note that $\epsilon_{\mathcal{U}}^{\mathcal{T}} \geq \rho_{\tau}(\epsilon_{\mathcal{U}}/2)$.

For $y \in \mathfrak{M}$ recall that $D_{\mathcal{F}}(y, \epsilon)$ is the *leafwise* closed ball of radius ϵ . Introduce a form of “leafwise Lebesgue number”, defined by

$$(5) \quad \epsilon_{\mathcal{U}}^{\mathcal{F}} = \min \{ \epsilon_{\mathcal{U}}^{\mathcal{F}}(y) \mid \forall y \in \mathfrak{M} \} \text{ , } \epsilon_{\mathcal{U}}^{\mathcal{F}}(y) = \sup \{ \epsilon \mid \forall y \in \mathfrak{M}, D_{\mathcal{F}}(y, \epsilon) \subset D_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/8) \}$$

Thus, for all $y \in \mathfrak{M}$, $D_{\mathcal{F}}(y, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset D_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/8)$. Note that for all $r > 0$ and $z' \in D_{\mathcal{F}}(z, \epsilon_{\mathcal{U}}^{\mathcal{F}})$, the triangle inequality implies that $B_{\mathfrak{M}}(z', r) \subset B_{\mathfrak{M}}(z, r + \epsilon_{\mathcal{U}}/8)$.

3. HOLONOMY OF FOLIATED SPACES

The holonomy pseudogroup of a foliated manifold (M, \mathcal{F}) generalizes the discrete cascade associated to a section of a flow. The holonomy pseudogroup for a matchbox manifold $(\mathfrak{M}, \mathcal{F})$ is defined analogously, although there are delicate issues of domains which must be considered.

A pair of indices (i, j) , $1 \leq i, j \leq \nu$, is said to be *admissible* if the *open* coordinate charts satisfy $U_i \cap U_j \neq \emptyset$. For (i, j) admissible, define $\mathfrak{D}_{i,j} = \pi_i(U_i \cap U_j) \subset \mathfrak{T}_i \subset \mathfrak{X}$. Then the closure $\overline{\mathfrak{D}_{i,j}} = \pi_i(\overline{U_i} \cap \overline{U_j})$. The hypotheses on foliation charts imply that plaques are either disjoint, or have connected intersection. This implies that there is a well-defined homeomorphism $h_{j,i}: \mathfrak{D}_{i,j} \rightarrow \mathfrak{D}_{j,i}$ with domain $D(h_{j,i}) = \mathfrak{D}_{i,j}$ and range $R(h_{j,i}) = \mathfrak{D}_{j,i}$. The map $h_{j,i}$ admits a continuous extension to $\overline{h_{j,i}}: \overline{\mathfrak{D}_{i,j}} \rightarrow \overline{\mathfrak{D}_{j,i}}$.

The maps $\mathcal{G}_{\mathcal{F}}^{(1)} = \{h_{j,i} \mid (i, j) \text{ admissible}\}$ are the transverse change of coordinates defined by the foliation charts. By definition they satisfy $h_{i,i} = Id$, $h_{i,j}^{-1} = h_{j,i}$, and if $U_i \cap U_j \cap U_k \neq \emptyset$ then $h_{k,j} \circ h_{j,i} = h_{k,i}$ on their common domain of definition. The *holonomy pseudogroup* $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is the topological pseudogroup modeled on \mathfrak{X} generated by compositions of the elements of $\mathcal{G}_{\mathcal{F}}^{(1)}$.

A sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ is *admissible*, if each pair $(i_{\ell-1}, i_\ell)$ is admissible for $1 \leq \ell \leq \alpha$, and the composition

$$(6) \quad h_{\mathcal{I}} = h_{i_\alpha, i_{\alpha-1}} \circ \dots \circ h_{i_1, i_0}$$

has non-empty domain. The domain $D(h_{\mathcal{I}})$ is the *maximal open subset* of $\mathfrak{D}_{i_0, i_1} \subset \mathfrak{T}_{i_0}$ for which the compositions are defined.

Given any open subset $U \subset D(h_{\mathcal{I}})$ we obtain a new element $h_{\mathcal{I}}|U \in \mathcal{G}_{\mathcal{F}}$ by restriction. Introduce

$$(7) \quad \mathcal{G}_{\mathcal{F}}^* = \{h_{\mathcal{I}}|U \mid \mathcal{I} \text{ admissible \& } U \subset D(h_{\mathcal{I}})\} \subset \mathcal{G}_{\mathcal{F}}.$$

The range of $g = h_{\mathcal{I}}|U$ is the open set $R(g) = h_{\mathcal{I}}(U) \subset \mathfrak{T}_{i_\alpha} \subset \mathfrak{X}$. Note that each map $g \in \mathcal{G}_{\mathcal{F}}^*$ admits a continuous extension $\overline{g}: \overline{D(g)} = \overline{U} \rightarrow \mathfrak{T}_{i_\alpha}$.

We introduce the standard notation for the orbits of the pseudogroup $\mathcal{G}_{\mathcal{F}}$, where for $w \in \mathfrak{X}$, set

$$(8) \quad \mathcal{O}(w) = \{g(w) \mid g \in \mathcal{G}_{\mathcal{F}}^*, w \in D(g)\} \subset \mathfrak{T}_*.$$

Given an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$, for each $0 \leq \ell \leq \alpha$, set $\mathcal{I}_\ell = (i_0, i_1, \dots, i_\ell)$ and

$$(9) \quad h_{\mathcal{I}_\ell} = h_{i_\ell, i_{\ell-1}} \circ \dots \circ h_{i_1, i_0}.$$

Given $\xi \in D(h_{\mathcal{I}})$ we adopt the notation $\xi_\ell = h_{\mathcal{I}_\ell}(\xi) \in \mathfrak{T}_{i_\ell}$. So $\xi_0 = \xi$ and $h_{\mathcal{I}}(\xi) = \xi_\alpha$.

Given $\xi \in D(h_{\mathcal{I}})$, let $x = x_0 = \tau_{i_0}(\xi_0) \in L_x$. Introduce the *plaque chain*

$$\mathcal{P}_{\mathcal{I}}(\xi) = \{\mathcal{P}_{i_0}(\xi_0), \mathcal{P}_{i_1}(\xi_1), \dots, \mathcal{P}_{i_\alpha}(\xi_\alpha)\}.$$

For each $0 \leq \ell < \alpha$, we have $\text{int}(\mathcal{P}_{i_\ell}(\xi_\ell)) \cap \text{int}(\mathcal{P}_{i_{\ell+1}}(\xi_{\ell+1})) \neq \emptyset$. Moreover, each $\mathcal{P}_{i_\ell}(\xi_\ell)$ is a strongly convex subset of the leaf L_x in the leafwise metric $d_{\mathcal{F}}$. Recall that $\mathcal{P}_{i_\ell}(x_\ell) = \mathcal{P}_{i_\ell}(\xi_\ell)$, so we also adopt the notation $\mathcal{P}_{\mathcal{I}}(x) \equiv \mathcal{P}_{\mathcal{I}}(\xi)$.

Intuitively, a plaque chain $\mathcal{P}_{\mathcal{I}}(\xi)$ is a sequence of successively overlapping convex “tiles” in L_0 starting at $x_0 = \tau_{i_0}(\xi_0)$, ending at $x_\alpha = \tau_{i_\alpha}(\xi_\alpha)$, and with each $\mathcal{P}_{i_\ell}(\xi_\ell)$ “centered” on the point $x_\ell = \tau_{i_\ell}(\xi_\ell)$.

3.1. Leafwise path holonomy. A *leafwise path* is a continuous map $\gamma: [0, 1] \rightarrow \mathfrak{M}$ with image in some leaf L of \mathcal{F} . The construction of the holonomy map h_γ associated to a leafwise path γ is a standard construction in foliation theory ([40], [31], [13], [14, Chapter 2]). We describe this in detail below, paying particular attention to domains and metric estimates.

Let \mathcal{I} be an admissible sequence. We say that (\mathcal{I}, w) *covers* γ , if there exists a partition $0 = s_0 < s_1 < \dots < s_\alpha = 1$ such that for the plaque chain $\mathcal{P}_{\mathcal{I}}(w) = \{\mathcal{P}_{i_0}(w_0), \mathcal{P}_{i_1}(w_1), \dots, \mathcal{P}_{i_\alpha}(w_\alpha)\}$ we have

$$(10) \quad \gamma([s_\ell, s_{\ell+1}]) \subset \text{int}(\mathcal{P}_{i_\ell}(w_\ell)), \quad 0 \leq \ell < \alpha, \quad \& \quad \gamma(1) \in \text{int}(\mathcal{P}_{i_\alpha}(w_\alpha))$$

It follows that $w_0 = \pi_{i_0}(\gamma(0)) \in D(h_{\mathcal{I}})$.

Now suppose we have two admissible sequences, $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ and $\mathcal{J} = (j_0, j_1, \dots, j_\beta)$, such that both (\mathcal{I}, w) and (\mathcal{J}, v) cover the leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$. Then

$$\gamma(0) \in \text{int}(\mathcal{P}_{i_0}(w_0)) \cap \text{int}(\mathcal{P}_{j_0}(v_0)) \quad , \quad \gamma(1) \in \text{int}(\mathcal{P}_{i_\alpha}(w_\alpha)) \cap \text{int}(\mathcal{P}_{j_\beta}(v_\beta))$$

Thus both (i_0, j_0) and (i_α, j_β) are admissible, and $v_0 = h_{j_0, i_0}(w_0)$, $w_\alpha = h_{i_\alpha, j_\beta}(v_\beta)$.

PROPOSITION 3.1. *The maps $h_{\mathcal{I}}$ and $h_{i_\alpha, j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0, i_0}$ agree on their common domains.* \square

3.2. Admissible sequences. Given a leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$, we next construct an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ with $w \in D(h_{\mathcal{I}})$ so that (\mathcal{I}, w) covers γ , and has “uniform domains”.

Inductively, choose a partition of the interval $[0, 1]$, $0 = s_0 < s_1 < \dots < s_\alpha = 1$ such that for each $0 \leq \ell \leq \alpha$, $\gamma([s_\ell, s_{\ell+1}]) \subset D_{\mathcal{F}}(x_\ell, \epsilon_{\mathcal{U}}^{\mathcal{F}})$ where $x_\ell = \gamma(s_\ell)$. As a notational convenience, we have let $s_{\alpha+1} = s_\alpha$, so that $\gamma([s_\alpha, s_{\alpha+1}]) = x_\alpha$. Note that we can choose $s_{\ell+1}$ to be the largest value of $s > s_\ell$ such that $d_{\mathcal{F}}(\gamma(s_\ell), \gamma(s_{\ell+1})) \leq \epsilon_{\mathcal{U}}^{\mathcal{F}}$. Thus, we can assume the estimate $\alpha \leq 1 + \|\gamma\|/\epsilon_{\mathcal{U}}^{\mathcal{F}}$ holds.

For each $0 \leq \ell \leq \alpha$, choose an index $1 \leq i_\ell \leq \nu$ so that $B_{\mathfrak{M}}(x_\ell, \epsilon_{\mathcal{U}}) \subset U_{i_\ell}$. Note that, for all $s_\ell \leq t \leq s_{\ell+1}$, $B_{\mathfrak{M}}(\gamma(t), \epsilon_{\mathcal{U}}/2) \subset U_{i_\ell}$, so that $x_{\ell+1} \in U_{i_\ell} \cap U_{i_{\ell+1}}$. It follows that $\mathcal{I}_\gamma = (i_0, i_1, \dots, i_\alpha)$ is an admissible sequence. Set $h_\gamma = h_{\mathcal{I}_\gamma}$. Then $h_\gamma(w) = w'$, where $w = \pi_{i_0}(x_0)$ and $w' = \pi_{i_\alpha}(x_\alpha)$.

The construction of the admissible sequence \mathcal{I}_γ above has an important special property. For $0 \leq \ell < \alpha$, note that $x_{\ell+1} \in D_{\mathcal{F}}(x_{\ell+1}, \epsilon_{\mathcal{U}}^{\mathcal{F}})$ implies that for some $s_\ell < s'_{\ell+1} < s_{\ell+1}$, we have that $\gamma([s'_{\ell+1}, s_{\ell+1}]) \subset D_{\mathcal{F}}(x_{\ell+1}, \epsilon_{\mathcal{U}}^{\mathcal{F}})$. Hence,

$$(11) \quad B_{\mathfrak{M}}(\gamma(t), \epsilon_{\mathcal{U}}/2) \subset U_{i_\ell} \cap U_{i_{\ell+1}} \quad , \quad \text{for all } s'_{\ell+1} \leq t \leq s_{\ell+1} \quad .$$

Then for all $s'_{\ell+1} \leq t \leq s_{\ell+1}$, the uniform estimate defining $\epsilon_{\mathcal{U}}^{\mathcal{F}} > 0$ in (4) implies that

$$(12) \quad B_{\mathfrak{X}}(\pi_{i_\ell}(\gamma(t)), \epsilon_{\mathcal{U}}^{\mathcal{T}}) \subset \mathfrak{D}_{i_\ell, i_{\ell+1}} \quad \& \quad B_{\mathfrak{X}}(\pi_{i_{\ell+1}}(\gamma(t)), \epsilon_{\mathcal{U}}^{\mathcal{T}}) \subset \mathfrak{D}_{i_{\ell+1}, i_\ell} \quad .$$

For the admissible sequence $\mathcal{I}_\gamma = (i_0, i_1, \dots, i_\alpha)$, recall that $x_\ell = \gamma(s_\ell)$ and we set $w_\ell = \pi_{i_\ell}(x_\ell)$. Then by the definition (6) of $h_{\mathcal{I}_\gamma}$ the condition (12) implies that $D_{\mathfrak{X}}(w_\ell, \epsilon_{\mathcal{U}}^{\mathcal{T}}) \subset D(h_\ell)$.

That is, $h_{\mathcal{I}_\gamma}$ is the composition of generators of $\mathcal{G}_{\mathcal{F}}^*$ which have uniform estimates on the radii of the metric balls contained in their domains, where $\epsilon_{\mathcal{U}}^{\mathcal{T}}$ is independent of γ .

There is a converse to the above construction, which associates to an admissible sequence a leafwise path. Let $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ be admissible, with corresponding holonomy map $h_{\mathcal{I}}$, and choose $w \in D(h_{\mathcal{I}})$ with $x = \tau_{i_0}(w)$.

For each $1 \leq \ell \leq \alpha$, recall that $\mathcal{I}_\ell = (i_0, i_1, \dots, i_\ell)$, and let $h_{\mathcal{I}_\ell}$ denote the corresponding holonomy map. For $\ell = 0$, let $\mathcal{I}_0 = (i_0, i_0)$. Note that $h_{\mathcal{I}_\alpha} = h_{\mathcal{I}}$ and $h_{\mathcal{I}_0} = \text{Id}: \mathfrak{T}_0 \rightarrow \mathfrak{T}_0$.

For each $0 \leq \ell \leq \alpha$, set $w_\ell = h_{\mathcal{I}_\ell}(w)$ and $x_\ell = \tau_{i_\ell}(w_\ell)$. By assumption, for $\ell > 0$, there exists $z_\ell \in \mathcal{P}_{\ell-1}(w_{\ell-1}) \cap \mathcal{P}_\ell(w_\ell)$.

Let $\gamma_\ell: [(\ell-1)/\alpha, \ell/\alpha] \rightarrow L_{x_0}$ be the leafwise piecewise geodesic segment from $x_{\ell-1}$ to z_ℓ to x_ℓ . Define the leafwise path $\gamma_{\mathcal{I}}^x: [0, 1] \rightarrow L_{x_0}$ from x_0 to x_α to be the concatenation of these paths. If we then cover $\gamma_{\mathcal{I}}^x$ by the charts determined by the given admissible sequence \mathcal{I} , it follows that $h_{\mathcal{I}} = h_{\gamma_{\mathcal{I}}^x}$.

Thus, given an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ and $w \in D(h_{\mathcal{I}})$ with $w' = h_{\mathcal{I}}(w)$, the choices above determine an initial chart φ_{i_0} with “starting point” $x = \tau_{i_0}(w) \in U_{i_0} \subset \mathfrak{M}$. Similarly, there is a terminal chart φ_{i_α} with “terminal point” $x' = \tau_{i_\alpha}(w') \in U_{i_\alpha} \subset \mathfrak{M}$. The leafwise path $\gamma_{\mathcal{I}}^x$ constructed above starts at x , ends at x' , and has image contained in the plaque chain $\mathcal{P}_{\mathcal{I}}(x)$.

On the other hand, if we start with a leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$, then the initial point $x = \gamma(a)$ and the terminal point $x' = \gamma(b)$ are both well-defined. However, there need not be a unique index j_0 such that $x \in U_{j_0}$ and similarly for the index j_β such that $x' \in U_{j_\beta}$. Thus, when one constructs an admissible sequence $\mathcal{J} = (j_0, \dots, j_\beta)$ from γ , the initial and terminal charts need not be well-defined. This was observed already in the proof of Proposition 3.1, which proved that

$$h_{\mathcal{I}}|U = h_{i_\alpha, j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0, i_0}|U \quad \text{for} \quad U = D(h_{\mathcal{I}}) \cap D(h_{i_\alpha, j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0, i_0}) \quad .$$

We introduce the following definition, which gives a uniform estimate of the effect of this ambiguity.

LEMMA 3.2. *There exists a continuous function $\kappa: \mathbb{R}_+ \rightarrow [1, \infty)$ such that for all admissible (i, j)*

$$(13) \quad d_{\mathfrak{X}}(h_{j,i}(w), h_{j,i}(w')) \leq \kappa(d_{\mathfrak{X}}(w, w')) \cdot d_{\mathfrak{X}}(w, w') \text{ , for all } w \neq w' \in \mathfrak{D}_{i,j} \text{ .}$$

Proof. For (i, j) admissible, the holonomy map $h_{j,i}$ extends to a homeomorphism of the closure of its domain, $\overline{h_{j,i}: \mathfrak{D}_{i,j} \rightarrow \mathfrak{D}_{j,i}}$. Thus, for $r > 0$, the product map $\overline{h_{j,i}} \times \overline{h_{j,i}}$ restricts to a homeomorphism of the compact set $\mathfrak{B}_{i,j}^r = \{(w, w') \mid w, w' \in \mathfrak{D}_{i,j} \text{ , } d_{\mathfrak{X}}(w, w') = r\}$. Define

$$(14) \quad \kappa(r) = \max \left\{ \frac{d_{\mathfrak{X}}(h_{j,i}(w), h_{j,i}(w'))}{r} \mid (i, j) \text{ admissible , } (w, w') \in \mathfrak{B}_{i,j}^r \right\} \text{ .}$$

Since $h_{i,i}$ is the identity map, the estimate (13) follows. \square

We conclude this discussion with a trivial observation, and an application which yields a key technical point, that the holonomy along a path is independent of “small deformations” of the path.

The observation is this. Let $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ be admissible, with associated holonomy map $h_{\mathcal{I}}$. Given $w, u \in D(h_{\mathcal{I}})$, then the germs of $h_{\mathcal{I}}$ at w and u admit a common extension, namely $h_{\mathcal{I}}$. Thus, if γ, γ' are leafwise paths defined as above from the plaque chains associated to (\mathcal{I}, w) and (\mathcal{I}, u) then the germinal holonomy maps along γ and γ' admit a common extension by Proposition 3.1. This is the basic idea behind the following technically useful result.

LEMMA 3.3. *Let $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ be leafwise paths. Suppose that $x = \gamma(0), x' = \gamma'(0) \in U_i$ and $y = \gamma(1), y' = \gamma'(1) \in U_j$. If $d_{\mathfrak{M}}(\gamma(t), \gamma'(t)) \leq \epsilon_U/4$ for all $0 \leq t \leq 1$, then the induced holonomy maps $h_\gamma, h_{\gamma'}$ agree on their common domain $D(h_\gamma) \cap D(h_{\gamma'}) \subset \mathfrak{T}_i$.*

3.3. Homotopy independence. Two leafwise paths $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ are homotopic if there exists a family of leafwise paths $\gamma_s: [0, 1] \rightarrow \mathfrak{M}$ with $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$. We are most interested in the special case when $\gamma(0) = \gamma'(0) = x$ and $\gamma(1) = \gamma'(1) = y$. Then γ and γ' are *endpoint-homotopic* if they are homotopic with $\gamma_s(0) = x$ for all $0 \leq s \leq 1$, and similarly $\gamma_s(1) = y$ for all $0 \leq s \leq 1$. Thus, the family of curves $\{\gamma_s(t) \mid 0 \leq s \leq 1\}$ are all contained in a common leaf L_x . The following property then follows from an inductive application of Lemma 3.3:

LEMMA 3.4. *Let $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ be endpoint-homotopic leafwise paths. Then their holonomy maps h_γ and $h_{\gamma'}$ agree on some open subset $U \subset D(h_\gamma) \cap D(h_{\gamma'}) \subset \mathfrak{T}_*$. In particular, they determine the same germinal holonomy maps.* \square

The following is another consequence of the total convexity of the plaques in the foliation covering:

LEMMA 3.5. *Suppose that $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ are leafwise paths for which $\gamma(0) = \gamma'(0) = x$ and $\gamma(1) = \gamma'(1) = x'$, and suppose that $d_{\mathfrak{M}}(\gamma(t), \gamma'(t)) < \epsilon_U/2$ for all $a \leq t \leq b$. Then $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ are endpoint-homotopic.* \square

Given $g \in \mathcal{G}_{\mathcal{F}}^*$ and $w \in D(g)$, let $[g]_w$ denote the germ of the map g at w . Set

$$(15) \quad \Gamma_{\mathcal{F}, w} = \{[g]_w \mid g \in \mathcal{G}_{\mathcal{F}}^* \text{ , } w \in D(g) \text{ , } g(w) = w\} \text{ .}$$

Given $x \in U_i$ with $w = \pi_i(x) \in \mathfrak{T}_*$, the elements of $\Gamma_{\mathcal{F}, w}$ form a group, and by Lemma 3.4 there is a well-defined homomorphism $h_{\mathcal{F}, x}: \pi_1(L_x, x) \rightarrow \Gamma_{\mathcal{F}, w}$ which is called the *holonomy group* of \mathcal{F} at x .

3.4. Non-trivial holonomy. Note that if $y \in L_x$ then the homomorphism $h_{\mathcal{F}, y}$ is conjugate (by an element of $\mathcal{G}_{\mathcal{F}}^*$) to the homomorphism $h_{\mathcal{F}, x}$. A leaf L is said to have *non-trivial germinal holonomy* if for some $x \in L$, the homomorphism $h_{\mathcal{F}, x}$ is non-trivial. If the homomorphism $h_{\mathcal{F}, x}$ is trivial, then we say that L_x is a *leaf without holonomy*. This property depends only on L , and not the basepoint $x \in L$. The foliated space \mathfrak{M} is said to be *without holonomy* if for every $x \in M$, the leaf L_x is without germinal holonomy.

LEMMA 3.6. *Let \mathfrak{M} be a foliated space without holonomy. Fix a regular covering for \mathfrak{M} as above. Let \mathcal{I}, \mathcal{J} be two plaques chains such that $w \in \text{Dom}(h_{\mathcal{I}}) \cap \text{Dom}(h_{\mathcal{J}})$ with $h_{\mathcal{I}}(w) = w' = h_{\mathcal{J}}(w)$. Then $h_{\mathcal{I}}$ and $h_{\mathcal{J}}$ have the same germinal holonomy at w . Thus, for each $w' \in \mathcal{O}(w)$ in the $\mathcal{G}_{\mathcal{F}}^*$ orbit of w , there is a well-defined holonomy germ $h_{w,w'}$.*

Proof. The composition $g = h_{\mathcal{J}}^{-1} \circ h_{\mathcal{I}}$ satisfies $g(w) = w$, so by assumption there is some open neighborhood $w \in U$ for which $g|_U$ is the trivial map. That is, $h_{\mathcal{I}}|_U = h_{\mathcal{J}}|_U$. \square

4. DYNAMICS OF MATCHBOX MANIFOLDS

We first recall several important classical definitions from topological dynamics, adapted to the case of matchbox manifolds. Then we recall several results concerning the dynamical properties of matchbox manifolds, which are established in the paper [18].

DEFINITION 4.1. *The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is equicontinuous if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $g \in \mathcal{G}_{\mathcal{F}}^*$, if $w, w' \in D(g)$ and $d_{\mathfrak{X}}(w, w') < \delta$, then $d_{\mathfrak{X}}(g(w), g(w')) < \epsilon$.*

DEFINITION 4.2. *The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is distal if for all $w, w' \in \mathfrak{T}_*$, if $w \neq w'$ then there exists $\delta_{w,w'} > 0$ such that for all $g \in \mathcal{G}_{\mathcal{F}}^*$ with $w, w' \in D(g)$, then $d_{\mathfrak{X}}(g(w), g(w')) \geq \delta_{w,w'}$.*

Distal and equicontinuous pseudogroups are closely related [5, 8, 23, 28, 33], while the following notion is their direct opposite.

DEFINITION 4.3. *The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is expansive, or more properly ϵ -expansive, if there exists $\epsilon > 0$ such that for all $w, w' \in \mathfrak{T}_*$, there exists $g \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in D(g)$ such that $d_{\mathfrak{X}}(g(w), g(w')) \geq \epsilon$.*

Equicontinuity for $\mathcal{G}_{\mathcal{F}}$ gives *uniform* control over the domains of arbitrary compositions of generators.

PROPOSITION 4.4. *Assume the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is equicontinuous. Then there exists $\delta_{\mathcal{U}}^{\mathcal{T}} > 0$ such that for every leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$, there is a corresponding admissible sequence $\mathcal{I}_{\gamma} = (i_0, i_1, \dots, i_{\alpha})$ so that $B_{\mathfrak{X}}(w_0, \delta_{\mathcal{U}}^{\mathcal{T}}) \subset D(h_{\mathcal{I}_{\gamma}})$, where $x = \gamma(0)$ and $w_0 = \pi_{i_0}(x)$.*

Moreover, for all $0 < \epsilon_1 \leq \epsilon_{\mathcal{U}}^{\mathcal{T}}$ there exists $0 < \delta_1 \leq \delta_{\mathcal{U}}^{\mathcal{T}}$ independent of the path γ , such that $h_{\mathcal{I}_{\gamma}}(D_{\mathfrak{X}}(w_0, \delta_1)) \subset D_{\mathfrak{X}}(w', \epsilon_1)$ where $w' = \pi_{i_{\alpha}}(\gamma(1))$.

Thus, $\mathcal{G}_{\mathcal{F}}^$ is equicontinuous as a family of local group actions.* \square

We next recall several results concerning minimal matchbox manifolds. First recall:

DEFINITION 4.5. *A foliated space \mathfrak{M} is minimal if each leaf $L \subset \mathfrak{M}$ is dense.*

The following is an immediate consequence of the definitions:

LEMMA 4.6. *A foliated space \mathfrak{M} is minimal if and only if for some regular covering of \mathfrak{M} , the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is minimal; that is, for all $w \in \mathfrak{T}_*$, the $\mathcal{G}_{\mathcal{F}}$ orbit $\mathcal{O}(w)$ of w is dense.*

The following result is, at first glance, very surprising. It has been previously shown for flows [1] and \mathbb{R}^n -actions [15]. The proof of it is given in detail in [18, Section 4.1], and fundamentally uses the conclusions of Proposition 4.4.

THEOREM 4.7. *If \mathfrak{M} is an equicontinuous matchbox manifold, then \mathfrak{M} is minimal.* \square

One of the main technical results of the work [18] is a much stronger version of Theorem 4.7, that not only is an equicontinuous matchbox manifold minimal, but it admits finite codings of its orbits with arbitrarily fine equivalence classes. The proof of the following is given in [18, Section 6].

THEOREM 4.8. *Let \mathfrak{M} is an equicontinuous matchbox manifold, and $w_0 \in \mathfrak{T}_*$ a basepoint. Then there exists a descending chain of clopen subsets*

$$\cdots \subset V_{\ell+1} \subset V_\ell \subset \cdots \subset V_0 \subset \mathfrak{T}_*$$

such that $w_0 \in V_\ell$ and $\text{diam}_{\mathfrak{X}}(V_\ell) < \delta_{\mathcal{U}}^T/2^\ell$ for all $\ell \geq 0$.

Moreover, each V_ℓ is $\mathcal{G}_{\mathcal{F}}$ -invariant. That is, if γ is a path with initial point $\gamma(0) \in V_\ell$, then the holonomy map h_γ satisfies $V_\ell \subset \text{Dom}(h_\gamma)$, and if $h_\gamma(V_\ell) \cap V_\ell \neq \emptyset$, then $h_\gamma(V_\ell) = V_\ell$. \square

It follows that the collection of subsets of \mathfrak{T}_* $\{h_\gamma(V_\ell) \mid \gamma(0) \in V_\ell\}$ forms a finite clopen partition of the transverse space \mathfrak{T}_* . Moreover, these sets are permuted by the local action of the holonomy pseudogroup.

The domains of arbitrary compositions of generators for the holonomy of an expansive foliation typically do not admit uniform estimates as in Proposition 4.4. However, one can give an estimate on the size of the domain of a particular map, based on the number of compositions used to define it. This result is fundamental for the proof of Theorem 1.2.

PROPOSITION 4.9. *For each $\epsilon > 0$ and $r > 0$, there exists $\delta(\epsilon, r) \geq \epsilon$ so that for any piecewise smooth leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$ with $\|\gamma\| \leq r$, then there exists an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ such that (\mathcal{I}, w) covers γ with:*

- (1) $w_0 = \pi_{i_0}(\gamma(0)) \in D(h_{\mathcal{I}})$ and $D_{\mathfrak{X}}(w_0, \delta(\epsilon, r)) \subset D(h_{\mathcal{I}})$;
- (2) $h_{\mathcal{I}}(D_{\mathfrak{X}}(w_0, \delta(\epsilon, r))) \subset D_{\mathfrak{X}}(w', \epsilon)$ where $w' = \pi_{i_\alpha}(\gamma(1))$.

Proof. By the arguments of Section 3.2, there exists an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ with $w \in D(h_{\mathcal{I}})$ and $\alpha \leq 1 + \|\gamma\|/\epsilon_{\mathcal{U}}^F \leq 1 + r/\epsilon_{\mathcal{U}}^F$ where $\epsilon_{\mathcal{U}}^F > 0$ is defined by (5).

We estimate the size of the domain $D(h_{\mathcal{I}})$. For each $0 \leq \ell \leq \alpha$, we have $\gamma([s_\ell, s_{\ell+1}]) \subset D_{\mathcal{F}}(x_\ell, \epsilon_{\mathcal{U}}^F)$ where $x_\ell = \gamma(s_\ell)$. Moreover, for the associated admissible sequence $\mathcal{I}_\gamma = (i_0, i_1, \dots, i_\alpha)$, we have that for all $0 \leq t \leq 1$, $B_{\mathfrak{M}}(\gamma(t), \frac{1}{2}\epsilon_{\mathcal{U}}) \subset U_{i_\ell}$.

The proof will be by downward induction. Let $w_\ell = \pi_{i_\ell}(x_\ell)$ and set $\mathcal{I}_\ell = (i_0, i_1, \dots, i_\ell)$ with corresponding holonomy map $h_{\mathcal{I}_\ell}$. Then $h_{\mathcal{I}_\ell}(w_0) = w_\ell$. Let $h_\ell = h_{i_{\ell+1}, i_\ell}$ so that $h_\ell \circ h_{\mathcal{I}_\ell} = h_{\mathcal{I}_{\ell+1}}$.

For every admissible pair (i, j) the holonomy homeomorphism $h_{j,i}$ is uniformly continuous as it has compact domain. Since there is only a finite number of distinct non-empty intersections $U_i \cap U_j$, for every $\epsilon > 0$ there exists a $0 < \delta_\epsilon \leq \epsilon$ such that for every admissible pair (i, j) if $w, w' \in D(h_{j,i})$ and $d_{\mathfrak{X}}(w, w') < \delta_\epsilon$ then $d_{\mathfrak{X}}(h_{j,i}(w), h_{j,i}(w')) < \epsilon$.

Recall $\epsilon_{\mathcal{U}}^T > 0$ as defined by (4). Given $\epsilon > 0$, if $\epsilon \geq \epsilon_{\mathcal{U}}^T$ set $\epsilon_\alpha = \epsilon_{\mathcal{U}}^T/2$, and otherwise set $\epsilon_\alpha = \epsilon$. Now proceed by downward induction. For $0 < \ell \leq \alpha$ assume that ϵ_ℓ has been defined. Then denote $\delta_\ell = \delta_{\epsilon_\ell}$ and $\epsilon_{\ell-1} = \delta_\ell$ as defined using equicontinuity as above.

By the choice of the covering of the admissible sequence \mathcal{I} , we have $D_{\mathfrak{X}}(w_\ell, \epsilon_{\mathcal{U}}^T) \subset D(h_\ell)$ and so $D_{\mathfrak{X}}(w_\ell, \delta_\ell) \subset D(h_\ell)$, and by the choice of δ_ℓ we have $D_{\mathfrak{X}}(w_\ell, \delta_\ell) \subset D(h_\ell \circ h_{\ell+1} \circ \cdots \circ h_\alpha)$. Then $\delta(\epsilon, r) = \delta_1$ satisfies the required conditions. \square

Finally, we recall a basic result of Epstein, Millet and Tischler [24] for foliated manifolds, whose proof applies verbatim in the case of foliated spaces.

THEOREM 4.10. *The union of all leaves without holonomy in a foliated space \mathfrak{M} is a dense G_δ subset of \mathfrak{M} . In particular, there exists at least one leaf without germinal holonomy. \square*

5. TRANSVERSE CANTOR FOLIATIONS

As noted in the introduction, the generalized solenoids with base an n -dimensional manifold, as in [26, 35, 41], and the tiling spaces associated to aperiodic tilings with finite local complexity [9, 11, 12, 19, 20] have a naturally given continuous family of local transversals to the foliation \mathcal{F} on \mathfrak{M} . In this section, we introduce the notion of a *Cantor foliation* \mathcal{H} *transverse to* \mathcal{F} , a concept which generalizes the properties of these transversals to an arbitrary matchbox manifold \mathfrak{M} .

Recall that we assume there is a fixed regular covering $\{U_i \mid 1 \leq i \leq \nu\}$ of \mathfrak{M} by foliation charts, as in Definition 2.7, with charts $\varphi_i: \overline{U}_i \rightarrow [-1, 1]^n \times \mathfrak{T}_i$ where $\mathfrak{T}_i \subset \mathfrak{X}$ is a clopen subset. By construction, each chart admits a foliated extension $\tilde{\varphi}_i: \tilde{U}_i \rightarrow (-2, 2)^n \times \mathfrak{T}_i$ where $\overline{U}_i \subset \tilde{U}_i \subset \mathfrak{X}$ is an open neighborhood of \overline{U}_i and $\tilde{\varphi}_i|_{\overline{U}_i} = \varphi_i$.

DEFINITION 5.1. *Let \mathfrak{M} be a matchbox manifold. We say that \mathfrak{M} admits a Cantor foliation \mathcal{H} transverse to \mathcal{F} if there exists an equivalence relation \approx on \mathfrak{M} such that:*

- (1) *for $x \in \mathfrak{M}$ the class $H_x = \{y \in \mathfrak{M} \mid y \approx x\}$ is a Cantor set;*
- (2) *for each $x \in U_i$ with $w = \pi_i(x) \in \mathfrak{T}_i$ there exists a clopen neighborhood $w \in V_x \subset \mathfrak{T}_i$ and a homeomorphism $\Phi_x: [-1, 1]^n \times V_x \rightarrow \tilde{U}_i$ such that, for each $\xi \in [-1, 1]^n$, the image $\Phi_x(\{\xi\} \times V_x) \subset \tilde{U}_i$ is a complete equivalence class.*

The leaves of the “foliation” \mathcal{H} are defined to be the equivalence classes of \approx .

We call V_x the model space for \mathcal{H} at x . For a standard foliation, the space V_x would be homeomorphic to $(-1, 1)^n$, while for a Cantor foliation, it is a homeomorphic to a Cantor set.

Condition 5.1.1 implies the leaves of \mathcal{H} are Cantor sets, and Condition 5.1.2 states that these leaves are “vertical” segments for a regular coordinate chart, after reparametrization by the maps Φ_x . In other words, every point $x \in \mathfrak{M}$ admits what is sometimes called in the foliation literature, a “bi-foliated neighborhood”, where the leaves of \mathcal{F} correspond to the “horizontal” Euclidean slices of $[-1, 1]^n \times V_x$, and the leaves of \mathcal{H} correspond to the “vertical” Cantor set slices.

The functions Φ_x are the adjustments to the local vertical foliation defined by the foliation coordinate system, so that the leaves of \mathcal{H} are coordinate independent, hence are globally defined. The requirement that the images of the maps Φ_x be allowed to take values in the open neighborhood \tilde{U}_i is due to the fact that on the boundary points of \overline{U}_i , the leaves of the foliation \mathcal{H} need not have constant horizontal coordinate λ_i .

6. FOLIATED MICROBUNDLES AND REEB STRUCTURE THEOREM

The Reeb Stability Theorem [13, 14, 40, 42] is one of the fundamental results of foliation theory. It states that for a compact leaf $L \subset M$ in a foliated manifold M with finite germinal holonomy group, there exists an open neighborhood $L \subset U$ so that U is a union of leaves of \mathcal{F} , and each leaf of $\mathcal{F}|_U$ is a finite covering of L . In particular, for a foliation defined by a flow, if the holonomy of a periodic orbit is finite, then nearby orbits are also periodic and have bounded length. This result has been generalized in various ways since then, with the most general version stated in terms of the concept of the “foliated microbundle” associated to the holonomy covering of leaf in a foliated space. See Milnor [37] for a discussion of the concept of foliated microbundles. In this section, we give the construction of the foliated microbundle associated to a leaf in a matchbox manifold, and introduce the spaces on which we construct a transverse foliation \mathcal{H} in the following sections.

Recall that we assume there is a fixed regular covering for \mathfrak{M} , as in Definition 2.7. Let $w_0 \in \text{int}(\mathfrak{T}_1)$ be a fixed base-point. Let $x_0 = \tau_1(w_0) \in U_1$ and L_0 be the leaf through x_0 . Let $h_{\mathcal{F}, x_0}: \pi_1(L_0, x_0) \rightarrow \mathcal{G}_{\mathcal{F}}^{w_0}$ denote the holonomy homomorphism of L_0 , whose kernel $\mathfrak{K}_0 \subset \pi_1(L_{x_0}, x_0)$ of $h_{\mathcal{F}, x_0}$ is a normal subgroup. Let $\Pi: \tilde{L}_0 \rightarrow L_0$ be the normal covering associated to \mathfrak{K}_0 and choose $\tilde{x}_0 \in \tilde{L}_0$ such that $\pi(\tilde{x}_0) = x_0$. By definition, given any closed path $\tilde{\gamma}: [0, 1] \rightarrow \tilde{L}_0$ with basepoint $\tilde{x}_0 = \tilde{\gamma}(0) = \tilde{\gamma}(1)$,

the image of $\tilde{\gamma}$ in L_0 has trivial germinal holonomy as a leafwise path in \mathfrak{M} . It follows that the holonomy map defined by a path $\tilde{\gamma}$ in \tilde{L}_0 starting at \tilde{x}_0 is determined by the endpoint $\tilde{\gamma}(1)$.

We next select a collection of points in \tilde{L}_0 which are sufficiently dense to capture all of the holonomy defined by the leaf L_0 .

DEFINITION 6.1. *Let (X, d_X) be a complete separable metric space. Given $0 < e_1 < e_2$, a subset $\mathcal{N} \subset X$ is a (e_1, e_2) -net (or Delone set) if:*

- (1) \mathcal{N} is e_1 -separated: for all $y \neq z \in \mathcal{N}$, $e_1 \leq d_X(y, z)$;
- (2) \mathcal{N} is e_2 -dense: for all $x \in X$, there exists some $z \in \mathcal{N}$ such that $d_X(x, z) \leq e_2$.

For example, given a leaf $L \subset \mathfrak{M}$, the intersection $L \cap \mathcal{T}$ is a countable set of points which satisfies the density condition (6.1.2) but need not satisfy the separation condition (6.1.1).

Our first step is to construct a net in the given leaf L_0 which satisfies the required net estimates for appropriately chosen e_1 and e_2 . It is elementary that given a separable, complete metric space X and any $e_2 > 0$, there exists $0 < e_1 < e_2$ and a (e_1, e_2) -net $\mathcal{N} \subset X$. Recall that $\epsilon_{\mathcal{U}}^{\mathcal{F}}$ defined by (5) was chosen so that every leafwise disk of radius $\epsilon_{\mathcal{U}}^{\mathcal{F}}$ is contained in a metric ball of \mathfrak{M} of radius $\epsilon_{\mathcal{U}}/2$. That is, for all $y \in \mathfrak{M}$, $D_{\mathcal{F}}(y, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset D_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/2)$.

Let $e_2 = \epsilon_{\mathcal{U}}^{\mathcal{F}}/4$, then choose $\mathcal{N}_0 \subset L_0$ an (e_1, e_2) -net for L_0 for some $0 < e_1 < \epsilon_{\mathcal{U}}^{\mathcal{F}}/4$. We can assume without loss of generality that $x_0 \in \mathcal{N}_0$. Condition (6.1.2) implies that the collection of leafwise open disks $\{B_{\mathcal{F}}(z, \epsilon_{\mathcal{U}}^{\mathcal{F}}/2) \mid z \in \mathcal{N}_0\}$ is a covering of L_0 .

For each $z \in \mathcal{N}_0$, choose an index $1 \leq i_z \leq \nu$ so that $B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$. Without loss, we can assume that $B_{\mathfrak{M}}(x_0, \epsilon_{\mathcal{U}}) \subset U_1$. Then note that for all $z' \in D_{\mathcal{F}}(z, \epsilon_{\mathcal{U}}^{\mathcal{F}})$, we have $z' \in B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}/2)$ so the triangle inequality implies that

$$(16) \quad D_{\mathcal{F}}(z', \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset B_{\mathfrak{M}}(z', \epsilon_{\mathcal{U}}/2) \subset B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$$

LEMMA 6.2. *The collection $\{U_{i_z} \mid z \in \mathcal{N}_0\}$ is a subcover for \mathfrak{M} , with Lebesgue number $\epsilon_{\mathcal{U}}/3$.*

Proof. Let $y \in \mathfrak{M}$, then L_0 is dense so there exists $y' \in L_0$ with $d_{\mathfrak{M}}(y, y') < \epsilon_{\mathcal{U}}/6$. Let $z \in \mathfrak{N}_0$ with $d_{\mathcal{F}}(y', z) < \epsilon_{\mathcal{U}}^{\mathcal{F}}/2$. Then $y' \in B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}/2)$ by (5), hence $y \in B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$ by (16). In fact, for all $y'' \in B_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/3)$ this shows that $y'' \in B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$ so that $B_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/3) \subset U_{i_z}$. \square

Let $\tilde{\mathcal{N}}_0 = \Pi^{-1}(\mathcal{N}_0)$ which is a (e_1, e_2) -net for \tilde{L}_0 with the Riemannian metric lifted from L_0 . The points of $\tilde{\mathcal{N}}_0$ are denoted by \tilde{z} , where $\Pi(\tilde{z}) = z \in \mathcal{N}_0$. In particular, $\tilde{x}_0 \in \tilde{\mathcal{N}}_0$ as $\Pi(\tilde{x}_0) = x_0 \in \mathcal{N}_0$.

For each $\tilde{z} \in \tilde{\mathcal{N}}_0$, let $z = \Pi(\tilde{z})$ and set $\tilde{U}_{\tilde{z}} = \overline{U}_{i_z} \times \{\tilde{z}\}$. For $(x, \tilde{z}) \in \tilde{U}_{\tilde{z}}$ define $\Pi: \tilde{U}_{\tilde{z}} \rightarrow \overline{U}_{i_z}$ by $\Pi(x, \tilde{z}) = x$. For $\tilde{z} \neq \tilde{z}' \in \tilde{\mathcal{N}}_0$ with $\Pi(\tilde{z}) = \Pi(\tilde{z}') = z$, the sets $\tilde{U}_{\tilde{z}}$ and $\tilde{U}_{\tilde{z}'}$ are disjoint by definition, though their projections to \mathfrak{M} agree.

For $\tilde{z} \in \tilde{\mathcal{N}}_0$ and $\tilde{y} = (x, \tilde{z}) \in \tilde{U}_{\tilde{z}}$, let $\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{y}) = \mathcal{P}_{i_z}(x) \times \{\tilde{z}\}$ denote the plaque of $\tilde{U}_{\tilde{z}}$ containing \tilde{y} . If $x \in \mathcal{P}_{i_z}(z)$ then we abuse notation and identify $\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{y})$ with the plaque of \tilde{L}_0 containing \tilde{z} . Note that $B_{\tilde{L}_0}(\tilde{z}, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset \tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z})$ for each $\tilde{z} \in \tilde{\mathcal{N}}_0$, so the collection $\{\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z}) \mid \tilde{z} \in \tilde{\mathcal{N}}_0\}$ is a covering of \tilde{L}_0 .

One thinks of the plaques $\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z})$ as “convex tiles”, and the collection $\{\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z}) \mid \tilde{z} \in \tilde{\mathcal{N}}_0\}$ as a “tiling” of \tilde{L}_0 . The interiors of the plaques need not be disjoint, so this is not a proper tiling in the usual sense (for example see [7, 9, 27], or [14, §11.3.C]).

The *foliated microbundle* of \tilde{L}_0 is the space

$$(17) \quad \tilde{\mathfrak{N}}_0 = \bigcup_{\tilde{z} \in \tilde{\mathcal{N}}_0} \tilde{U}_{\tilde{z}} / \sim$$

where $\tilde{y} \in \tilde{U}_{\tilde{z}}$ and $\tilde{y}' \in \tilde{U}_{\tilde{z}'}$ are identified if $\Pi(\tilde{y}) = \Pi(\tilde{y}')$ and $\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z}) \cap \tilde{\mathcal{P}}_{\tilde{z}'}(\tilde{z}') \neq \emptyset$.

For each $\tilde{z} \in \tilde{\mathcal{N}}_0$, let $\mathfrak{T}_{\tilde{z}} = \mathfrak{T}_{i_{\tilde{z}}} \times \{\tilde{z}\}$. The composition $\tilde{\varphi}_{\tilde{z}} \equiv \varphi_{i_{\tilde{z}}} \circ \Pi: \tilde{U}_{\tilde{z}} \rightarrow [-1, 1]^n \times \mathfrak{T}_{\tilde{z}}$ defines a coordinate chart on $\tilde{\mathfrak{M}}_0$, making it into a foliated space with foliation denoted by $\tilde{\mathcal{F}}$. Let $\tilde{\pi}_{\tilde{z}}: \tilde{U}_{\tilde{z}} \rightarrow \mathfrak{T}_{\tilde{z}}$ be the normal coordinate, and $\tilde{\lambda}_{\tilde{z}}: \tilde{U}_{\tilde{z}} \rightarrow [-1, 1]^n$ be the leafwise coordinate.

Given $\tilde{z} \in \tilde{\mathcal{N}}_0$, subset $V \subset \mathfrak{T}_{\tilde{z}}$ and $\xi \in [-1, 1]^n$, we obtain a local section for $\tilde{\mathcal{F}}$ by

$$(18) \quad \tilde{\tau}_{\tilde{z}, \xi}: V \rightarrow \tilde{U}_{\tilde{z}}, \quad \tilde{\tau}_{\tilde{z}, \xi}(w) = \tilde{\varphi}_{\tilde{z}}^{-1}(\xi, w) = (\varphi_{i_{\tilde{z}}}^{-1}(\xi, w), \tilde{z})$$

The usefulness of the foliated microbundle concept as constructed, is that it provides a uniform setting for all of the holonomy maps for paths in the leaf \tilde{L}_0 .

A path $\tilde{\gamma}: [0, 1] \rightarrow \tilde{L}_0$ is said to be *nice*, if there exists a partition $a = s_0 < s_1 < \dots < s_\alpha = b$ such that for each $0 \leq \ell \leq \alpha$, the restriction $\tilde{\gamma}: [s_\ell, s_{\ell+1}] \rightarrow \tilde{L}_0$ is a geodesic segment between points $\tilde{z}_\ell = \tilde{\gamma}(s_\ell), \tilde{z}_{\ell+1} = \tilde{\gamma}(s_{\ell+1}) \in \tilde{\mathcal{N}}_0$ with $d_{\mathcal{F}}(\tilde{z}_\ell, \tilde{z}_{\ell+1}) < \epsilon_{\mathcal{U}}^{\mathcal{F}}$. Then $\mathcal{I} = (i_{\tilde{z}_0}, \dots, i_{\tilde{z}_\alpha})$ is an admissible sequence for *both* $\tilde{\mathcal{F}}$ and \mathcal{F} , and so defines holonomy maps $\tilde{h}_{\mathcal{I}}$ for $\tilde{\mathcal{F}}$ and $h_{\mathcal{I}}$ for \mathcal{F} . Clearly, $\tilde{h}_{\mathcal{I}}$ is just the lift of $h_{\mathcal{I}}$, and $h_{\mathcal{I}}$ is the holonomy map for the leafwise path $\gamma = \Pi \circ \tilde{\gamma}$ constructed in Section 3. As before, we note that $\tilde{h}_{\mathcal{I}}$ depends only on the endpoints of \mathcal{I} . For $\tilde{z} \in \tilde{\mathcal{N}}_0$ let $\tilde{h}_{\tilde{z}}$ denote the holonomy along some nice path $\tilde{\gamma}_{\tilde{z}}$ from \tilde{x}_0 to \tilde{z} , considered as a transformation of the space $\tilde{\mathfrak{T}}$, which is the disjoint union of the local transversals $\mathfrak{T}_{\tilde{z}}$. Let $h_{\tilde{z}}$ denote the holonomy along the projected path, $\gamma_{\tilde{z}} = \Pi \circ \tilde{\gamma}_{\tilde{z}}$.

If \mathfrak{M} be an equicontinuous matchbox manifold, then by Theorem 4.8, for any $\epsilon > 0$, there exists a $\mathcal{G}_{\mathcal{F}}$ -invariant clopen subset $w_0 \in V \subset \mathfrak{T}_*$ satisfying $\text{diam}_{\mathfrak{X}}(V) < \epsilon$. For $h_\gamma \in \mathcal{G}_{\mathcal{F}}^*$ with $V \subset \text{Dom}(h_\gamma)$, set $V_\gamma = h_\gamma(V)$. For $\tilde{z} \in \tilde{\mathcal{N}}_0$ there is a nice path $\gamma_{\tilde{z}}$ from \tilde{z}_0 to \tilde{z} which defines a holonomy map denoted by $h_{\tilde{z}} \equiv h_{\gamma_{\tilde{z}}}$. Then for $\tilde{z} \in \tilde{\mathcal{N}}_0$ define

$$(19) \quad V_{\tilde{z}} = h_{\tilde{z}}(V) \subset \mathfrak{T}_{i_{\tilde{z}}}, \quad \tilde{V}_{\tilde{z}} = \tilde{h}_{\tilde{z}}(V) = V_{\tilde{z}} \times \{\tilde{z}\} \subset \mathfrak{T}_{i_{\tilde{z}}}.$$

The union of the sets $V_{\tilde{z}}$ is the saturation of V under the action of the pseudogroup $\mathcal{G}_{\mathcal{F}}^*$, and hence it forms a clopen partition of \mathfrak{T}_* . Introduce the local coordinate chart saturations of these sets:

$$(20) \quad \mathfrak{U}_{\tilde{z}}^V = \pi_{i_{\tilde{z}}}^{-1}(V_{\tilde{z}}) \subset \overline{U}_{i_{\tilde{z}}}, \quad \tilde{\mathfrak{U}}_{\tilde{z}}^V = \mathfrak{U}_{\tilde{z}}^V \times \{\tilde{z}\} \subset \tilde{U}_{i_{\tilde{z}}}.$$

Then $\mathfrak{U}_{\tilde{z}}^V$ is the union of the plaques in $\overline{U}_{i_{\tilde{z}}}$ through the points of $V_{\tilde{z}}$.

DEFINITION 6.3. *The Thomas tube associated with V is the subset of the microbundle $\tilde{\mathfrak{M}}_0$,*

$$(21) \quad \tilde{\mathfrak{N}}(V) = \bigcup_{\tilde{z} \in \tilde{\mathcal{N}}_0} \tilde{\mathfrak{U}}_{\tilde{z}}^V$$

The image $\Pi(\tilde{\mathfrak{N}}(V)) \subset \mathfrak{M}$ is the saturation by \mathcal{F} of the clopen set V , hence $\Pi(\tilde{\mathfrak{N}}(V)) = \mathfrak{M}$. Note that each leaf \tilde{L} of $\tilde{\mathcal{F}}$ in $\tilde{\mathfrak{N}}(V)$ has no holonomy and is properly embedded by construction, though the projection L of \tilde{L} in \mathfrak{M} is recurrent by minimality.

The assertion of Theorem 1.1 follows if we show that for V with sufficiently small diameter, then there exists a transverse Cantor foliation on $\tilde{\mathfrak{N}}(V)$ which defines a fibration map $\tilde{\mathfrak{N}}(V) \rightarrow \tilde{L}_0$ and is equivariant with respect to the map $\Pi: \tilde{\mathfrak{N}}(V) \rightarrow \mathfrak{M}$.

Next, let $L_0 \subset \mathfrak{M}$ be a leaf, and $\tilde{K} \subset \tilde{L}$ a connected compact subset of the holonomy covering $\Pi: \tilde{L}_0 \rightarrow L_0$, and set $K = \Pi(\tilde{K})$. We assume that \tilde{K} is a union of plaques for convenience. Assume also that the restriction of the covering map, $\Pi: \tilde{K} \rightarrow L_0$, is an injection. Then the set $\tilde{K} \cap \tilde{\mathcal{N}}$ is finite, and the image $\mathcal{N}_K = \Pi(\tilde{K} \cap \tilde{\mathcal{N}}) \subset \mathfrak{T}_*$ is a finite set of points. Choose a basepoint $\tilde{z}_0 \in (\tilde{K} \cap \tilde{\mathcal{N}})$ and set $z_0 = \Pi(\tilde{z}_0)$. We say that a clopen neighborhood $z_0 \in V \subset \mathfrak{T}_*$ is \tilde{K} -disjoint if the images

$$\left\{ V_{\tilde{z}} \equiv h_{\tilde{z}}(V) \mid \tilde{z} \in \tilde{K} \cap \tilde{\mathcal{N}} \right\}$$

are disjoint. The Reeb neighborhood of K is then defined by

$$(22) \quad \mathfrak{N}(K, V) \equiv \Pi(\tilde{\mathfrak{N}}(\tilde{K}, V)) \quad , \quad \tilde{\mathfrak{N}}(\tilde{K}, V) = \bigcup_{\tilde{z} \in \tilde{K} \cap \tilde{\mathcal{N}}} \tilde{\pi}_{\tilde{z}}^{-1}(V_{\tilde{z}}) \subset \tilde{\mathfrak{M}}_0$$

Note that each leaf of the restricted foliation $\tilde{\mathcal{F}}|_{\tilde{\mathfrak{N}}(\tilde{K}, V)}$ is a properly embedded compact subset, and the holonomy $\tilde{h}_{\tilde{\gamma}}$ along any closed loop $\tilde{\gamma}$ contained in $\tilde{\mathfrak{N}}(\tilde{K}, V)$ is trivial. Then the same holds for each path component of $\mathfrak{N}(K, V)$. Thus, the dynamics of \mathcal{F} restricted to $\mathfrak{N}(K, V)$ is trivial.

The assertion of Theorem 1.2 is that for V with sufficiently small diameter, there exists a transverse Cantor foliation \mathcal{H} on $\mathfrak{N}(K, V)$ which defines a fibration map $\mathfrak{N}(K, V) \rightarrow K$.

The spaces $\tilde{\mathfrak{N}}(V)$ in (21) and $\tilde{\mathfrak{N}}(K, V)$ in (22) are the unions of coordinate charts for the foliation $\tilde{\mathcal{F}}$ on $\tilde{\mathfrak{N}}_0$. Each of these covering charts naturally has a product structure, and thus a transverse foliation, but these product structures need not coincide on overlaps of the charts. The construction of a global transverse foliation requires that these local transverse foliations be “perturbed” to match up on overlaps, so that they define a global transverse foliation. As the phrase goes, this is easier said than done. That is, we must give coordinate independent definitions of transversals in each flow box. For this purpose, we next introduce leafwise Voronoi tessellations and Delaunay simplicial complexes, and study their properties.

PART II - FOLIATED VORONOI AND DELAUNAY STRUCTURES

The concept of a Voronoi cell decomposition (or tessellation) of the plane, or of a Euclidean space more generally, is extraordinarily useful for applications of geometry to a variety of problems, and is thus very well-studied. For a nice historical discussion of this concept, and a good discussion of such applications, see the Introduction of the book [39]. In Part II, we develop the basic concepts of Voronoi tessellations and Delaunay triangulations, as needed for our application studying the topological structure of matchbox manifolds.

7. VORONOI TESSELLATIONS

Let \mathfrak{M} be a matchbox manifold, and $L \subset \mathfrak{M}$ a leaf with induced leafwise Riemannian metric d_L . We recall the notion of a *Delaunay set* for L .

DEFINITION 7.1. *Given $0 < d_1 < d_2$, a subset $\mathcal{M} \subset L$ is a (d_1, d_2) -net (or Delaunay set) if:*

- (1) \mathcal{M} is d_1 -separated: for all $y \neq z \in \mathcal{M}$, $d_1 \leq d_L(y, z)$;
- (2) \mathcal{M} is d_2 -dense: for all $x \in L$, there exists some $z \in \mathcal{M}$ such that $d_L(x, z) \leq d_2$.

Given a net $\mathcal{M} \subset L$, one can associate to it the Voronoi tessellation of L , which is a partition of the space into compact star-like regions, called cells. The cells can be thought of as a tiling of L , although there is no assumption that there is only a finite number of isometry types of cells. Thus, we are considering a very general form of tiling.

Let $\lambda_{\mathcal{F}} > 0$ be a constant such that for all $x \in L$, the closed disk $D_L(x, \lambda_{\mathcal{F}}) \subset L$ is strongly convex.

We assume there is given a (d_1, d_2) -net, $\mathcal{M} \subset L$, which satisfies $d_2 \leq \lambda_{\mathcal{F}}/5$

Introduce the “leafwise nearest-neighbor distance” function

$$(23) \quad \kappa_L(y) = \inf \{d_L(x, y) \mid x \in \mathcal{M}\}$$

Note that $\kappa_L(y) = 0$ if and only if $y \in \mathcal{M}$. For $x \in \mathcal{M}$, define

$$(24) \quad \mathfrak{C}(x) = \{y \in L \mid d_L(x, y) = \kappa_L(y)\}$$

That is, $\mathfrak{C}(x)$ consists of the points $y \in L$ which are closer to x in the leafwise metric than to any other point of \mathcal{M} . We say that $\mathfrak{C}(x)$ is a *Dirichlet region* or *Voronoi cell* in L defined by the net \mathcal{M} .

LEMMA 7.2. *For each $x \in \mathcal{M}$, $D_L(x, d_1/2) \subset \mathfrak{C}(x) \subset B_L(x, d_2)$. In particular, $\mathfrak{C}(x)$ has diameter at most $2d_2$.*

Proof. Note that for all $y \in L$, we have $\kappa_L(y) \leq d_2$ as there exists $z \in \mathcal{M}$ such that $d_L(y, z) \leq d_2$. Hence, for $x \in \mathcal{M}$, $\mathfrak{C}(x) \subset B_L(x, d_2)$. On the other hand, for all $x \neq y \in \mathcal{M}$ we have $x \notin B_L(y, d_1)$ and thus $B_L(x, d_1/2) \cap B_L(y, d_1/2) = \emptyset$. The inclusion $D_L(x, d_1/2) \subset \mathfrak{C}(x)$ follows as a result. \square

LEMMA 7.3. *For each $x \in \mathcal{M}$, the set $\mathfrak{C}(x)$ is star-like with respect to x .*

Proof. Let $y \in \mathfrak{C}(x)$ and let $\sigma_{x,y}: [0, 1] \rightarrow D_L(x, \lambda_{\mathcal{F}}) \subset L$ be the geodesic segment with $\sigma_{x,y}(0) = x$ and $\sigma_{x,y}(1) = y$. Fix $0 < s < 1$ and let $z = \sigma_{x,y}(s)$. We must show that $z \in \mathfrak{C}(x)$.

Let $u \in \mathcal{M}$ with $u \neq x$, then $d_L(y, x) \leq d_L(y, u)$. The strong convexity of $B_L(x, \lambda_{\mathcal{F}})$ implies that $d_L(x, z) = s \cdot d_L(x, y)$ and $d_L(z, y) = (1 - s) \cdot d_L(x, y)$. Then the triangle inequality implies that

$$d_L(z, u) \geq d_L(y, u) - d_L(z, y) = d_L(y, u) - (1 - s) \cdot d_L(x, y) \geq d_L(y, x) - (1 - s) \cdot d_L(x, y)$$

so $d_L(x, z) \leq d_L(u, z)$. \square

The following is an immediate consequence of the definitions and Lemma 7.3, and implies that $\mathcal{C}(\mathcal{M})$ is a “generalized” tiling of the metric space L .

PROPOSITION 7.4. *The collection $\mathcal{C}(\mathcal{M}) = \{\mathfrak{C}(y) \mid y \in \mathcal{M}\}$ defines a closed covering of L . The interiors of the cells are disjoint, star-like open sets.* \square

Given $x \in \mathcal{M}$, introduce the *vertex-sets*

$$(25) \quad V(x) = \{y \in \mathcal{M} \mid \mathfrak{C}(y) \cap \mathfrak{C}(x) \neq \emptyset\}; \quad V_*(x) = \{y \in V(x) \mid y \neq x\}$$

Note that $V(x)$ is a finite set by the net condition on \mathcal{M} , and $y \in V_*(x)$ if and only if $x \in V_*(y)$. The “star-neighborhood” of $\mathfrak{C}(x)$ is the set

$$(26) \quad \mathfrak{S}(x) = \bigcup_{y \in V(x)} \mathfrak{C}(y)$$

LEMMA 7.5. *For each $x \in \mathcal{M}$, $\mathfrak{S}(x) \subset B_L(x, 3d_2) \subset B_L(x, \lambda_{\mathcal{F}})$, hence $\mathfrak{S}(x)$ is contained in a strongly convex subset of L_x .*

Proof. Suppose that $\mathfrak{C}(x) \cap \mathfrak{C}(y) \neq \emptyset$. As $\mathfrak{C}(y)$ has diameter at most $2\lambda_{\mathcal{F}}/5$, the claim follows. \square

Next we investigate the structure of the Voronoi cells $\mathfrak{C}(x)$. For $y \in V_*(x)$ set

$$(27) \quad H(x, y) = \{z \in D_L(x, \lambda_{\mathcal{F}}) \mid d_L(x, z) \leq d_L(y, z)\}$$

One thinks of each $H(x, y)$ as the “half-plane” defined by the set of points in the closed disk $D_L(x, \lambda_{\mathcal{F}})$ which are closer to x than to y . Clearly, each $H(x, y)$ is closed, and the total convexity of $D_L(x, \lambda_{\mathcal{F}})$ implies that

$$\mathfrak{C}(x) = \bigcap_{y \in V_*(x)} H(x, y)$$

Note that $y \in V_*(x)$ if and only if the intersection

$$L(x, y) = H(x, y) \cap H(y, x) \neq \emptyset$$

For example, if L is isometric to Euclidean space \mathbb{R}^2 , then indeed $H(x, y)$ is a half-plane restricted to the disk $D_L(x, \lambda_{\mathcal{F}})$, and $L(x, y)$ is a line segment. In general, one has:

LEMMA 7.6. *Let $y \in V_*(x)$, then $L(x, y)$ is a codimension-one closed submanifold.*

Proof. We have $x, y \in D_L(x, \lambda_{\mathcal{F}})$ and the metric d_L is strongly convex when restricted to $D_L(x, \lambda_{\mathcal{F}})$. Thus the functions $f_x(z) = d_L(x, z)^2$ and $f_y(z) = d_L(y, z)^2$ are both regular on $D_L(x, \lambda_{\mathcal{F}})$ away from x and y , which implies that $L(x, y)$ is a codimension-one closed submanifold. \square

Let $V_r(x) \subset V_*(x)$ be the subset corresponding to the open faces of the boundary of $\mathfrak{C}(x)$. That is, $y \in V_r(x)$ if and only if the closed face $\partial_y \mathfrak{C}(x) = \mathfrak{C}(x) \cap L(x, y)$ has non-trivial interior, as a subset of the submanifold $L(x, y)$. Then the topological boundary $\partial \mathfrak{C}(x)$ is the finite union

$$(28) \quad \partial \mathfrak{C}(x) = \bigcup_{y \in V_r(x)} \partial_y \mathfrak{C}(x)$$

8. DELAUNAY SIMPLICIAL COMPLEX

We next introduce the *Delaunay simplicial complex* of L obtained from a net \mathcal{M} , using the “inscribed sphere” characterization of the simplices.

Let $0 < r < \lambda_{\mathcal{F}}$. Then the leafwise sphere of radius r centered at z is

$$S_L(z, r) \equiv \{y \in L \mid d_L(z, y) = r\} = D_L(z, r) - B_L(z, r)$$

Note that if $B_L(x, r) \cap \mathcal{M} = \emptyset$ then $r \leq d_2$ by the definition of d_2 .

The Delaunay complex $\Delta(\mathcal{M})$ of L derived from the net \mathcal{M} is defined by specifying the subsets of \mathcal{M} which form the vertices of the simplices in $\Delta(\mathcal{M})$. Denote by $\Delta^{(k)}(\mathcal{M})$ the collection of k -simplices.

DEFINITION 8.1. *For each $z_0 \in \mathcal{M}$, the set $\Delta(z_0) = \{z_0\}$ is a 0-simplex in $\Delta^{(0)}(\mathcal{M})$.*

For $k > 0$, a $(k+1)$ -tuple $\{z_0, \dots, z_k\} \subset \mathcal{M}$ forms a k -simplex $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{M})$ if there exists $z \in L$ and $0 < r \leq d_2$ such that $B_L(z, r) \cap \mathcal{M} = \emptyset$, and $\{z_0, \dots, z_k\} \subset S_L(z, r) \cap \mathcal{M}$.

If $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{M})$, then every subset of $(\ell+1)$ -points, $\{z_{i_0}, \dots, z_{i_\ell}\} \subset \{z_0, \dots, z_k\}$, yields an ℓ -simplex $\Delta(z_{i_0}, \dots, z_{i_\ell}) \in \Delta^{(\ell)}(\mathcal{M})$, as the inscribed sphere condition holds for all subsets. In particular, we have well-defined face and boundary operators defined on $\Delta(\mathcal{M})$.

Note that if the manifold L is Euclidean, then given a k -simplex $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{M})$, the convex hull of the set of vertices defines a geometric k -simplex in L . For a non-Euclidean manifold, one must choose a procedure for “filling in” the geometric simplex spanned by a set of vertices. For a 1-simplex $\Delta(z_0, z_1)$, this can be done canonically using the geodesic between z_1 and z_0 , which is unique due to the strong convexity of $B_L(z_0, \lambda_{\mathcal{F}})$. For the higher-dimensional simplices, our approach is to inductively fill in the faces using the geodesic cone from each successive vertex. If the manifold L is not flat, then the resulting geometric simplex will typically depend on the ordering of the vertices. As this construction is used several times in the following, we give it explicitly.

Define the standard k -simplex Δ^k in \mathbb{R}^{k+1} by the barycentric coordinate approach,

$$\Delta^k = \{(t_0, \dots, t_k) \mid t_\ell \geq 0, t_0 + \dots + t_k = 1\}$$

The vertices of Δ^k are the coordinate vectors $\vec{e}_\ell = (0, \dots, 1, \dots, 0)$ where the unique non-zero entry 1 is located in the $\ell+1$ coordinate position.

LEMMA 8.2. *Let $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{M})$ be given, so that $\{z_0, \dots, z_k\} \subset B_L(z_0, \lambda_{\mathcal{F}})$. Then there exists a diffeomorphism $\sigma_k: \Delta^k \rightarrow L$ such that $\sigma_k(\vec{e}_\ell) = z_\ell$, and the maps σ_k are natural with respect to the face operators.*

Proof. The map σ_k is defined inductively on the faces of Δ^k . First, set $\sigma_k(\vec{e}_\ell) = z_\ell$.

Given a string $I = i_0 < i_1 < \dots < i_\nu$ with $0 \leq i_0$ and $i_\nu \leq k$, define the I -face $\partial_I \Delta^k$ to be the subset consisting of points where the only non-zero entries are in the coordinates appearing in the string. For $0 < \nu$, let $I' = i_0 < i_1 < \dots < i_{\nu-1}$. By induction, we may assume that the map $\sigma_k: \partial_{I'} \Delta^k \rightarrow L$ has been defined.

Note that each point $\vec{v} \in \partial_I \Delta^k$ can be written as $(1-s) \cdot \vec{v}' + s \cdot \vec{e}_{i_\nu}$ where $\vec{v}' \in \partial_{I'} \Delta^k$ and $0 \leq s \leq 1$. The point $z' = \sigma_k(\vec{v}') \in L$ is defined by the inductive hypothesis, and so there exists a unique geodesic segment $\tau: [0, 1] \rightarrow B_L(z_0, \lambda_{\mathcal{F}})$ such that $\tau(0) = z_{i_\nu}$ and $\tau(1) = z'$. Then set $\sigma_k(\vec{v}) = \tau(s)$. The resulting map on all of Δ^k satisfies the requirements of the Lemma. \square

The Voronoi cell decomposition and Delaunay triangulation of L are closely related. For Euclidean space, one says that $\Delta(\mathcal{M})$ is dual to the Voronoi tessellation. For the general case of a Riemannian manifold with bounded geometry, this statement becomes problematic, as discussed for example in [21, 32]. Here is a basic result, whose proof follows almost immediately from the definitions.

PROPOSITION 8.3. *For $z_0 \in \mathcal{M}$, let $\{z_1, \dots, z_k\} \subset V_r(z_0)$. Then $L(z_0, z_1) \cap \dots \cap L(z_0, z_k) \neq \emptyset$ if and only if $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{M})$.*

Proof. Let $x \in L(z_0, z_1) \cap \dots \cap L(z_0, z_k)$ and set $r = d_L(x, z_0)$. Then $d_L(x, z_i) = d_L(x, z_0) = r$ for each $1 \leq i \leq k$. Thus, $\{z_0, \dots, z_k\} \subset S_L(z, r)$ and by the definition of the Voronoi cells, $B_L(x, r) \cap \mathcal{M} = \emptyset$. This implies $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{M})$.

Conversely, if $\{z_0, \dots, z_k\} \subset S_L(x, r)$ and $B_L(x, r) \cap \mathcal{M} = \emptyset$, then $x \in L(z_0, z_j)$ for all $1 \leq j \leq k$ hence $L(z_0, z_1) \cap \dots \cap L(z_0, z_k) \neq \emptyset$ \square

A point $x \in \partial \mathfrak{C}(z_0)$ is *extremal* if the distance function $d_L(z_0, y)$ has a local maximum on $\mathfrak{C}(z_0)$ at $y = x$. For $z_i \in V_r(z_0)$, the boundary component $\partial_{z_i} \mathfrak{C}(z_0) = \mathfrak{C}(z_0) \cap L(z_0, z_i)$ has codimension one, thus a point $x \in \partial \mathfrak{C}(z_0)$ is extremal exactly when there is $\{z_1, \dots, z_n\} \subset V_r(z_0)$ with

$$\omega(z_0, \dots, z_n) = L(z_0, z_1) \cap \dots \cap L(z_0, z_n)$$

and $\omega(z_0, \dots, z_n)$ is the center of an inscribed sphere containing $\{z_0, \dots, z_n\}$ with radius

$$r(z_0, \dots, z_n) = d_{\mathcal{F}}(z_\ell, \omega(z_0, \dots, z_n)) , \quad 0 \leq \ell \leq n$$

It is possible that $\Delta(\mathcal{M})$ contains $(n+1)$ -simplices, where some collection of $(n+1)$ -hyperplanes satisfy $L(z_0, z_1) \cap \dots \cap L(z_0, z_{n+1}) \neq \emptyset$. This is a degenerate condition, as typically every collection of $(n+1)$ -hyperplanes in $D_L(z_0, \lambda_{\mathcal{F}})$ should have empty intersection. This motivates the definition:

DEFINITION 8.4. *The simplicial complex $\Delta(\mathcal{M})$ is regular if $\Delta^{(n+1)}(\mathcal{M}) = \emptyset$. We say that the net \mathcal{M} is regular if $\Delta(\mathcal{M})$ is regular.*

Much of the technical work in the following is to give conditions on a regular net \mathcal{M} , such that for a net \mathcal{M}' sufficiently close to \mathcal{M} , then $\Delta(\mathcal{M}')$ will again be regular.

9. FOLIATED VORONOI STRUCTURE

We extend the construction of the Voronoi cell decomposition and the dual Delaunay triangulation for a single leaf, to a “parametrized version” which applies uniformly to the leaves of a smooth matchbox manifold \mathfrak{M} . This requires the choice of a uniform regular transversal for \mathcal{F} which satisfies appropriate regularity hypotheses.

A closed subset $\mathcal{X} \subset \mathfrak{M}$ is a *standard transversal* for \mathcal{F} if $\mathcal{X} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_{p_*}$ where for each $1 \leq \ell \leq p_*$ there exists a foliation chart φ_{i_ℓ} , closed subset $X_\ell \subset \mathfrak{T}_{i_\ell}$ and basepoint $v_\ell \in (-1, 1)^n$ such that

$$\mathcal{X}_\ell = \varphi_{i_\ell}^{-1}(v_\ell, X_\ell)$$

The transversal $\mathcal{X} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_{p_*}$ is *invariant* if for every leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$ with $z_0 = \gamma(0) \in \mathcal{X}_j$ and $z_1 = \gamma(1) \in \mathcal{X}_k$ the induced holonomy map satisfies $h_\gamma(X_j) = X_k$.

Let $\mathfrak{N} \subset \mathfrak{M}$ be a subset which is a union of coordinate plaques as in (21) or (22).

The transversal \mathcal{X} is (d_1, d_2) -*uniform* on a subset $\mathfrak{N} \subset \mathfrak{M}$ if there exists $0 < d_1 < d_2 \leq \lambda_{\mathcal{F}}/5$ such that for each $x \in \mathfrak{N}$, with K_x the path connected component of \mathfrak{N} containing x , then the intersection $\mathcal{M}(K_x) = K_x \cap \mathcal{X}$ is a (d_1, d_2) -net for the metric space K_x with the leafwise metric $d_{\mathcal{F}}$. If $\mathfrak{N} = \mathfrak{M}$ then we let $\mathcal{M}(x) = \mathcal{M}(L_x) = L_x \cap \mathcal{X}$.

The transversal \mathcal{X} is *regular uniform* for \mathfrak{N} if it is (d_1, d_2) -uniform for some (d_1, d_2) , and for each $x \in \mathfrak{N}$, the Delaunay simplicial complex $\Delta(\mathcal{M}(K_x))$ associated to $\mathcal{M}(K_x)$ is regular.

Assume there is given a regular uniform invariant transversal \mathcal{X} for \mathfrak{N} . The nearest-neighbor distance function κ_L extends to a leafwise function

$$(29) \quad \kappa_{\mathcal{F}}(y) = \inf \{d_{\mathcal{F}}(x, y) \mid x \in \mathcal{X}\}$$

For $x \in \mathcal{X}$ define

$$(30) \quad \mathfrak{C}(\mathfrak{N}, x) = \{y \in L_x \cap \mathfrak{N} \mid d_{\mathcal{F}}(x, y) = \kappa_{\mathcal{F}}(y)\}$$

That is, $\mathfrak{C}(\mathfrak{N}, x)$ is the Voronoi cell in $L_x \cap \mathfrak{N}$ defined by the net $\mathcal{M}(x)$, and consists of the points $y \in L_x \cap \mathfrak{N}$ which are closer to x in the leafwise metric than to any other point of $\mathcal{M}(x)$.

If $\mathfrak{N} = \mathfrak{M}$, then we suppress the notation \mathfrak{N} . For example, we write $\mathfrak{C}(x) = \mathfrak{C}(\mathfrak{M}, x)$.

For $1 \leq \ell \leq p_*$ define the *Voronoi cylinder* by $\mathfrak{C}_{\ell}(\mathfrak{N}) = \bigcup_{x \in \mathcal{X}_{\ell}} \mathfrak{C}(\mathfrak{N}, x)$. We obtain the *Voronoi decomposition* $\mathfrak{N} = \mathfrak{C}_0(\mathfrak{N}) \cup \dots \cup \mathfrak{C}_{p_*}(\mathfrak{N})$ associated to \mathcal{X} .

For each cylinder $\mathfrak{C}_{\ell}(\mathfrak{N})$ define its “star-neighborhood” $\mathfrak{S}_{\ell}(\mathfrak{N}) = \bigcup_{x \in \mathcal{X}_{\ell}} \mathfrak{S}(\mathfrak{N}, x)$. We assume that $\mathfrak{S}_{\ell}(\mathfrak{N}) \subset U_{i_{\ell}}$ for each $1 \leq \ell \leq p_*$.

A transversal \mathcal{X} is said to be *nice* if it is regular, uniform, invariant and satisfies this condition.

Now let $\mathcal{X} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_{p_*}$ be a nice transversal for $\mathfrak{N} \subset \mathfrak{M}$.

Define the leafwise simplicial complex $\Delta_{\mathcal{F}}(\mathcal{X})$ associated to the nice transversal \mathcal{X} as follows. A k -simplex $\Delta(z_0, \dots, z_k) \in \Delta_{\mathcal{F}}(\mathcal{X})$ if $\{z_0, \dots, z_k\} \subset L_{z_0} \cap \mathcal{X}$, and there exists $z \in L_{z_0}$ and $r > 0$ such that $B_{\mathcal{F}}(z, r) \cap \mathcal{X} = \emptyset$, and $\{z_0, \dots, z_k\} \subset S_{\mathcal{F}}(z, r) \cap \mathcal{X}$. Thus, $\Delta(z_0, \dots, z_k) \in \Delta(\mathcal{M}(z_0))$.

Let $\Delta(z_0, \dots, z_k) \in \Delta_{\mathcal{F}}(\mathcal{X})$ be given. Let $1 \leq j_0, \dots, j_k \leq p_*$ be indices such that $z_{\ell} \in \mathcal{X}_{j_{\ell}}$ for $0 \leq \ell \leq k$. Then we have that $z_0 \in \mathcal{X}_{j_0} \subset U_{i_{j_0}}$ so $z_0 \in \mathcal{P}_{i_{j_0}}(z_0)$. The assumption that \mathcal{X} is normal and the condition $d_1/2 \leq r \leq d_2 < \lambda_{\mathcal{F}}^*$ implies that we also have $z_{\ell} \in \mathcal{P}_{i_{j_0}}(z_0)$ for $1 \leq \ell \leq k$, as the plaque has radius $\lambda_{\mathcal{F}}^*$. That is, all of the vertices of the simplex $\Delta(z_0, \dots, z_k)$ are contained in the same plaque. In particular, this implies that for $\ell \neq \ell'$ the sets $\mathcal{X}_{j_{\ell}}$ are $\mathcal{X}_{j_{\ell'}}$ are disjoint.

Also recall that the star neighborhood $\mathfrak{S}_{k_0}(\mathfrak{N}) \subset U_{i_{j_0}}$, so for each $1 \leq \ell \leq n$, we have that $\mathcal{X}_{j_{\ell}} \subset U_{i_{j_0}}$.

Let $z'_0 \in \mathcal{X}_{k_0}$. Let $\mathcal{P}_{i_{k_0}}(z'_0)$ denote the plaque of $U_{i_{k_0}}$ containing z'_0 . For each $1 \leq \ell \leq n$, let $z'_{\ell} = \mathcal{X}_{i_{\ell}} \cap \mathcal{P}_{i_{k_0}}(z'_0)$ be the unique point of $\mathcal{X}_{i_{\ell}}$ contained in the plaque defined by z'_0 . Observe that the points z'_{ℓ} depend continuously on $z'_0 \in \mathcal{X}_{k_0}$.

DEFINITION 9.1. *A nice transversal \mathcal{X} is stable if for each k -simplex $\Delta(z_0, \dots, z_k) \in \Delta_{\mathcal{F}}(\mathcal{X})$ and $z'_0 \in \mathcal{X}_{k_0}$, we have $\Delta(z'_0, \dots, z'_k) \in \Delta_{\mathcal{F}}(\mathcal{X})$.*

PROPOSITION 9.2. *Let \mathcal{X} be a nice stable transversal for $\mathfrak{N} \subset \mathfrak{M}$. Then \mathcal{X} induces a transverse foliation \mathcal{H} on \mathfrak{N} .*

Proof. Let $\Delta(z_0, \dots, z_n) \in \Delta_{\mathcal{F}}(\mathcal{X})$ be given, with $z_{\ell} \in \mathcal{X}_{i_{\ell}}$. Then as $\{z_0, \dots, z_n\} \subset B_{\mathcal{F}}(z_0, \lambda_{\mathcal{F}}^*)$ we have $\ell \neq \ell'$ implies that $i_{\ell} \neq i_{\ell'}$. Without loss of generality, we assume that $\ell < \ell'$ implies $i_{\ell} < i_{\ell'}$.

For each $z'_0 \in \mathcal{X}_{i_0}$ let $z'_{\ell} = \mathcal{P}_{i_0}(z'_0) \cap \mathcal{X}_{i_{\ell}}$. The stable hypothesis implies that $\Delta(z'_0, \dots, z'_n) \in \Delta_{\mathcal{F}}(\mathcal{X})$.

By Lemma 8.2 there exists a geodesic filling map $\sigma_{n, z'_0}: \Delta^n \rightarrow \mathcal{P}_{i_0}(z'_0) \subset L_{z'_0}$ associated to $\Delta(z'_0, \dots, z'_n)$ which is natural with respect to the face maps.

For $\vec{v} \in \Delta^n$ and $z'_0, z''_0 \in \mathcal{X}_{i_0}$ we identify $\sigma_{n, z'_0}(\vec{v}) \approx \sigma_{n, z''_0}(\vec{v})$. This equivalence relation defines the leaves of \mathcal{H} .

Every point $x \in \mathfrak{N}$ lies in the image $\sigma_{n, z_0}(\Delta^n)$ for some $\Delta(z_0, \dots, z_n) \in \Delta_{\mathcal{F}}(\mathcal{X})$, as $\Delta_{\mathcal{F}}(\mathcal{X})$ defines a triangulation of each leaf of $\mathcal{F}|_{\mathfrak{N}}$. Then by the naturality property of the geodesic filling maps, the equivalence relation \approx is well-defined on all of \mathfrak{N} . \square

We make some remarks about this construction. First, for each $1 \leq \ell \leq p_*$ and $z, z' \in \mathcal{X}_\ell$ then $z \approx z'$. That is, each transversal \mathcal{X}_ℓ is a “leaf” of the foliation \mathcal{H} on \mathfrak{N} defined by \approx .

Second, for each 1-simplex $\Delta(z_0, z_1) \in \Delta_{\mathcal{F}}(\mathcal{X})$ the equivalence relation \approx identifies points with the same barycentric coordinate on the unique geodesic ray joining z'_1 to z'_0 where $\Delta(z'_0, z'_1)$ is the transverse transport of the given 1-simplex. Thus, \approx is uniquely defined on the 1-skeleton of the leafwise triangulation $\Delta_{\mathcal{F}}(\mathcal{X})$. If \mathcal{F} is an orientable foliation by 1-dimensional leaves, that is, it is defined by a flow, then we are done, and the equivalence relation \approx depends canonically on the choice of the uniform transversal \mathcal{X} , but is independent of its ordering.

If the leaves of \mathcal{F} have dimension $n > 1$, then the “spanning geodesic procedure” in the proof of Lemma 8.2 may well-depend upon the ordering of the vertices in each simplex. However, this local ordering of the vertices in simplices is determined by the choice of the transversal $\mathcal{X} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_{p_*}$. Thus \approx is well-defined, given the choice of the transversal \mathcal{X} with its ordering.

It remains to show the existence of a nice stable transversal \mathcal{X} for $\mathfrak{N} \subset \mathfrak{M}$. Our approach is to develop effective estimates on the process of defining the Voronoi tessellation for a net and its associated Delaunay triangulation.

PART III - EUCLIDEAN STRUCTURES

In Part III, we consider the special case of \mathbb{R}^n with the standard Euclidean metric $d_{\mathbb{R}^n}$ and associated norm $\|\cdot\|$. The definition of the Delaunay simplicial complex in Definition 8.1 can then be expressed in terms of linear equations and estimates. We establish several key technical estimates on the center and radius of an inscribed sphere, in terms of the defining points in a net.

10. DELAUNAY SIMPLICES IN \mathbb{R}^n

We consider each $\vec{x} \in \mathbb{R}^n$ as a column vector, and let $\vec{x} \bullet \vec{y} = \vec{x}^t \cdot \vec{y}$ denote the “dot-product” of two vectors, where \vec{x}^t denotes the matrix transpose of \vec{y} , and $\vec{x}^t \cdot \vec{y}$ denotes the standard matrix product.

Given a collection of n vectors, $\{\vec{a}_1, \dots, \vec{a}_n\} \subset \mathbb{R}^n$, let \mathbf{A} denote the $n \times n$ matrix with these vectors as rows. Denote the operator norm for \mathbf{A} by

$$\|\mathbf{A}\| = \max \{ \|\mathbf{A} \cdot \vec{x}\| \mid \vec{x} \in \mathbb{R}^n, \|\vec{x}\| = 1 \}$$

The Cauchy-Schwartz inequality then yields the estimate

$$(31) \quad \|\mathbf{A}\|^2 \leq \|\vec{a}_1\|^2 + \dots + \|\vec{a}_n\|^2$$

We assume given a collection of vectors $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$ which admit an inscribed sphere with center $\omega(\vec{y}_0, \dots, \vec{y}_n)$ and radius $r(\vec{y}_0, \dots, \vec{y}_n)$. For each $1 \leq k \leq n$, set $\vec{u}_k = (\vec{y}_{k-1} - \vec{y}_n)$. Then $\|\vec{u}_k\| \leq 2r(\vec{y}_0, \dots, \vec{y}_n)$ as all vectors \vec{y}_i are contained in a set with diameter $2r(\vec{y}_0, \dots, \vec{y}_n)$.

Let \mathbf{U} denote the $n \times n$ matrix whose rows are the transposes of the vectors \vec{u}_k . Let $|\mathbf{U}|$ denote the absolute value of the determinant, which equals the volume of the parallelepiped in \mathbb{R}^n with edges $\{\vec{u}_1, \dots, \vec{u}_n\}$, or equal to $n!$ times the volume of the simplex in \mathbb{R}^n with vertices $\{\vec{y}_0, \dots, \vec{y}_n\}$.

Suppose there exists constants $0 < e_1 < e_2$ and $\varepsilon, \delta > 0$ such that

- (1) $e_1 \leq \|\vec{y}_i - \vec{y}_j\|$ for all $0 \leq i \neq j \leq n$
- (2) $e_1/2 \leq r(\vec{y}_0, \dots, \vec{y}_n) \leq e_2$ and hence $\|\vec{y}_i - \vec{y}_j\| \leq 2e_2$
- (3) $|\mathbf{U}| \geq \delta$

Our goal is to obtain effective estimates for the inscribed sphere for a small perturbation of a given net, so also assume we are given vectors $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$ such that

- (4) $\|\vec{y}_i - \vec{z}_i\| < \varepsilon$ for all $0 \leq i \leq n$

We determine values of the constants $\varepsilon, \delta > 0$ such that the points $\{\vec{z}_0, \dots, \vec{z}_n\}$ admit a unique inscribed sphere with center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ and radius $r(\vec{z}_0, \dots, \vec{z}_n)$, and obtain estimates for

$$\|\omega(\vec{z}_0, \dots, \vec{z}_n) - \omega(\vec{y}_0, \dots, \vec{y}_n)\| \quad \text{and} \quad |r(\vec{z}_0, \dots, \vec{z}_n) - r(\vec{y}_0, \dots, \vec{y}_n)|.$$

The hyperplanes $L(\vec{y}_{k-1}, \vec{y}_n)$, for $1 \leq k \leq n$, are described by the equations

$$\begin{aligned} L(\vec{y}_{k-1}, \vec{y}_n) &= \{\vec{x} \in \mathbb{R}^n \mid (\vec{y}_{k-1} - \vec{y}_n) \bullet (\vec{x} - (\vec{y}_{k-1} + \vec{y}_n)/2) = 0\} \\ &= \{\vec{x} \in \mathbb{R}^n \mid \vec{u}_k \bullet \vec{x} = 1/2 \cdot \vec{u}_k \bullet \vec{u}_k + \vec{u}_k \bullet \vec{y}_n\} \\ (32) \quad &= \left\{ \xi + \vec{y}_n \mid \vec{u}_k \bullet \xi = 1/2 \cdot \|\vec{u}_k\|^2 \right\} \end{aligned}$$

where $\xi = \vec{x} - \vec{y}_n$ represents the coordinates for $L(\vec{y}_k, \vec{y}_n)$ with \vec{y}_n translated to the origin. The center $\vec{\omega}(\vec{y}_0, \dots, \vec{y}_n) \in \mathbb{R}^n$ is thus the solution of the system of equations

$$(33) \quad \mathbf{U} \cdot \xi = \frac{1}{2} \cdot \vec{\lambda}(\mathbf{U}) \quad \text{so} \quad \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n) = \frac{1}{2} \cdot \left\{ \mathbf{U}^{-1} \cdot \vec{\lambda}(\mathbf{U}) \right\} + \vec{y}_n$$

where $\vec{\lambda}(\mathbf{U}) = (\|\vec{u}_1\|^2, \dots, \|\vec{u}_n\|^2)^t$ is the column vector with entries $\|\vec{u}_k\|^2$.

Similarly, let \mathbf{V} denote the $n \times n$ matrix whose rows are the transposes of the vectors $\vec{v}_k = \vec{z}_{k-1} - \vec{z}_n$ for $1 \leq k \leq n$, and set $\vec{\lambda}(\mathbf{V}) = (\|\vec{v}_1\|^2, \dots, \|\vec{v}_n\|^2)^t$. Assuming that \mathbf{V}^{-1} exists, then for $\zeta = \vec{x} - \vec{z}_n$, the solution of the matrix equation

$$(34) \quad \mathbf{V} \cdot \zeta = \frac{1}{2} \cdot \vec{\lambda}(\mathbf{V}) \quad , \quad \vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) = \frac{1}{2} \cdot \left\{ \mathbf{V}^{-1} \cdot \vec{\lambda}(\mathbf{V}) \right\} + \vec{z}_n$$

is the center for an inscribed sphere containing the points $\{\vec{z}_0, \dots, \vec{z}_n\}$.

Our next goal is to obtain an effective estimate on $\|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\|$. This will be based upon obtaining effective estimates for the matrix norms $\|\mathbf{U}\|^{-1}$ and $\|\mathbf{V}^{-1}\|$.

Let $\mathbf{W} = \mathbf{V} - \mathbf{U}$ so $\mathbf{V} = \mathbf{U} + \mathbf{W}$, and set $\mathbf{Q} = \mathbf{W}\mathbf{U}^{-1}$.

LEMMA 10.1. *Assume that $\|\mathbf{Q}\| \leq 1/2$, then \mathbf{V}^{-1} exists, and $\|\mathbf{V}^{-1}\| \leq 2\|\mathbf{U}^{-1}\|$.*

Proof. Since $\mathbf{V} = (\mathbf{I} + \mathbf{Q})\mathbf{U}$, and we assume that \mathbf{U}^{-1} exists and $\|\mathbf{Q}\| < 1$, its inverse is given by

$$(37) \quad \mathbf{V}^{-1} = \mathbf{U}^{-1}(\mathbf{I} + \mathbf{Q})^{-1} = \mathbf{U}^{-1}(\mathbf{I} - \mathbf{Q} + \mathbf{Q}^2 - \mathbf{Q}^3 \pm \dots)$$

Hence, estimating the norm of the infinite sum using the triangle inequality inductively, we obtain

$$(38) \quad \|\mathbf{V}^{-1}\| \leq \|\mathbf{U}^{-1}\| \cdot \|\mathbf{I} - \mathbf{Q} + \mathbf{Q}^2 - \mathbf{Q}^3 \pm \dots\| \leq \|\mathbf{U}^{-1}\|/(1 - \|\mathbf{Q}\|)$$

As $1/(1 - \|\mathbf{Q}\|) \leq 2$ this completes the proof. \square

Next, the triangle inequality and our given data yield the following estimates, where $e_3 = e_2 + \varepsilon$,

$$(39) \quad \|\vec{v}_k - \vec{u}_k\| \leq \|\vec{z}_{k-1} - \vec{y}_{k-1}\| + \|\vec{z}_n - \vec{y}_n\| \leq 2\varepsilon$$

$$(38) \quad e_1 - 2\varepsilon \leq \|\vec{u}_k\| - \|\vec{v}_k - \vec{u}_k\| \leq \|\vec{v}_k\| \leq \|\vec{u}_k\| + \|\vec{v}_k - \vec{u}_k\| \leq 2e_2 + 2\varepsilon = 2e_3$$

$$(39) \quad \left| \|\vec{v}_k\|^2 - \|\vec{u}_k\|^2 \right| = |(\vec{v}_k - \vec{u}_k) \bullet (\vec{v}_k + \vec{u}_k)| \leq \|\vec{v}_k - \vec{u}_k\| \cdot (\|\vec{v}_k\| + \|\vec{u}_k\|) \leq 4\varepsilon(e_2 + e_3)$$

We then have the matrix norm estimate,

$$(40) \quad \|\mathbf{W}\| = \|\mathbf{V} - \mathbf{U}\| \leq \sqrt{\|\vec{v}_1 - \vec{u}_1\|^2 + \dots + \|\vec{v}_n - \vec{u}_n\|^2} \leq 2\varepsilon\sqrt{n}$$

We next estimate the norm $\|\mathbf{U}^{-1}\|$. Our colleague Shmuel Friedland suggested the use of the *Hadamard determinantal inequality* in the proof of the following general estimate:

LEMMA 10.2. *Let \mathbf{A} be an $n \times n$ -matrix whose determinant has absolute value $|\mathbf{A}| > 0$, and such that each column of \mathbf{A} has norm at most C . Then*

$$(41) \quad \|\mathbf{A}^{-1}\| \leq n \cdot C^{n-1}/|\mathbf{A}|$$

Proof. For an invertible $n \times n$ matrix \mathbf{C} , let $0 < |\sigma_n(\mathbf{C})| \leq \dots \leq |\sigma_1(\mathbf{C})|$ denote the singular values of \mathbf{C} , ordered by their norms. Recall that $\|\mathbf{C}\|^2 = \|\mathbf{C}^t \cdot \mathbf{C}\| = |\sigma_1(\mathbf{C})|^2$.

Thus, $\|\mathbf{A}^{-1}\| = 1/|\sigma_1(\mathbf{A}^{-1})| = 1/|\sigma_n(\mathbf{A})|$ where $|\sigma_n(\mathbf{A})| > 0$ is the smallest singular value of \mathbf{A} .

Let $\mathbf{adj}(\mathbf{A})$ denote the adjoint of \mathbf{A} . Since $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{adj}(\mathbf{A})$ it follows that the singular values of $\mathbf{adj}(\mathbf{A})$ are all the $(n-1)$ products of the n singular values of \mathbf{A} . Hence the largest singular value for $\mathbf{adj}(\mathbf{A})$ is

$$\sigma_1(\mathbf{adj}(\mathbf{A})) = \sigma_1(\mathbf{A}) \cdots \sigma_{n-1}(\mathbf{A})$$

Each entry of $\mathbf{adj}(\mathbf{A})$ is an $(n-1)$ minor of \mathbf{A} , and thus its absolute value is less or equal to C^{n-1} by the Hadamard determinantal inequality.

Now if $\mathbf{B} = [b_{ij}] \in \mathbb{R}^{n \times n}$ is such that the absolute value of each entry is bounded above by $\alpha > 0$, then $\|\mathbf{B}\| \leq n\alpha$, since each L^2 -norm of the column of \mathbf{B} is bounded by $\alpha\sqrt{n}$ and we apply (31).

Thus $|\sigma_1(\mathbf{adj}(\mathbf{A}))| = |\sigma_1(\mathbf{A}) \cdots \sigma_{n-1}(\mathbf{A})| \leq n \cdot C^{n-1}$, and the claim (41) follows. \square

COROLLARY 10.3. *Let $\{\vec{u}_1, \dots, \vec{u}_n\} \subset \mathbb{R}^n$ satisfy $\|\vec{u}_k\| \leq 2e_2$ for $1 \leq k \leq n$, and $|\mathbf{U}| \geq \delta$. Then*

$$(42) \quad \|\mathbf{U}^{-1}\| \leq n(2e_2)^{n-1}/|\mathbf{U}| \leq n \cdot (2e_2)^{n-1}/\delta$$

The estimates (40) and (42) yield

$$(43) \quad \begin{aligned} \|\mathbf{Q}\| &= \|\mathbf{W} \cdot \mathbf{U}^{-1}\| \leq \|\mathbf{W}\| \cdot \|\mathbf{U}^{-1}\| \leq \{2\varepsilon\sqrt{n}\} \cdot \|\mathbf{U}^{-1}\| \\ &\leq \{2\varepsilon\sqrt{n}\} \cdot \{n(2e_2)^{n-1}/\delta\} = \varepsilon \cdot 2^n n^{3/2} (e_2)^{n-1}/\delta \end{aligned}$$

COROLLARY 10.4. *Assume that $\varepsilon < \delta/2^{n+1}n^{3/2}(e_2)^{n-1}$, then $\|\mathbf{Q}\| < 1/2$ and so \mathbf{V}^{-1} exists. Moreover, we have the estimate $\|\mathbf{V}^{-1}\| \leq n \cdot 2^n (e_2)^{n-1}/\delta$.*

Proof. This follows from Lemma 10.1 and Corollary 10.3. \square

We next estimate the remaining terms in the equations (33) and (34). By (38) and (39),

$$(44) \quad \|\vec{\lambda}(\mathbf{V})\| \leq (2e_3)^2, \quad \|\vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U})\| \leq 4\varepsilon(e_2 + e_3)$$

We now return to the task of estimating $\|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\|$. Consider:

$$(45) \quad \begin{aligned} &2 \|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\| \\ &= \left\| \left\{ \mathbf{V}^{-1} \cdot \vec{\lambda}(\mathbf{V}) \right\} + 2\vec{z}_n - \left\{ \mathbf{U}^{-1} \cdot \vec{\lambda}(\mathbf{U}) \right\} - 2\vec{y}_n \right\| \\ &= \left\| 2(\vec{z}_n - \vec{y}_n) + \left\{ \mathbf{U}^{-1}(\mathbf{I} + \mathbf{Q})^{-1} \cdot \vec{\lambda}(\mathbf{V}) \right\} - \left\{ \mathbf{U}^{-1} \cdot \vec{\lambda}(\mathbf{U}) \right\} \right\| \\ &\leq 2\|\vec{z}_n - \vec{y}_n\| + \|\mathbf{U}^{-1}\| \cdot \left\| (\mathbf{I} - \mathbf{Q} + \mathbf{Q}^2 - \mathbf{Q}^3 \pm \dots) \cdot \vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U}) \right\| \\ &\leq 2\|\vec{z}_n - \vec{y}_n\| + \|\mathbf{U}^{-1}\| \cdot \left\{ \|\vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U})\| + \|(-\mathbf{Q} + \mathbf{Q}^2 - \mathbf{Q}^3 \pm \dots) \cdot \vec{\lambda}(\mathbf{V})\| \right\} \\ &\leq 2\|\vec{z}_n - \vec{y}_n\| + \|\mathbf{U}^{-1}\| \cdot \left\{ \|\vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U})\| + \|\vec{\lambda}(\mathbf{V})\| \cdot \|\mathbf{Q}\|/(1 - \|\mathbf{Q}\|) \right\} \end{aligned}$$

Assume that $\varepsilon < \delta/2^{n+1}n^{3/2}(e_2)^{n-1}$, hence $\|\mathbf{Q}\| < 1/2$ by Corollary 10.4 and so $\|\mathbf{Q}\|/(1 - \|\mathbf{Q}\|) < 1$ and thus \mathbf{V}^{-1} exists. We use the more accurate estimate $\|\mathbf{Q}\| \leq \varepsilon 2^n n^{3/2} (e_2)^{n-1}/\delta$ from (43), which combined with the previous estimates $\|\vec{y}_n - \vec{z}_n\| < \varepsilon$, (42) and (44), then (45) becomes

$$\begin{aligned} &2 \|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\| \\ &\leq 2\|\vec{z}_n - \vec{y}_n\| + \|\mathbf{U}^{-1}\| \cdot \left\{ \|\vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U})\| + \|\vec{\lambda}(\mathbf{V})\| \cdot \|\mathbf{Q}\|/(1 - \|\mathbf{Q}\|) \right\} \\ &\leq 2\varepsilon + \{n \cdot (2e_2)^{n-1}/\delta\} \cdot \left\{ 4\varepsilon(e_2 + e_3) + (2e_3)^2 \cdot 2\varepsilon \cdot 2^n n^{3/2} (e_2)^{n-1}/\delta \right\} \end{aligned}$$

Then using that $e_3 = e_2 + \varepsilon > e_2$ we have

$$(46) \quad \|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\| < \varepsilon \cdot \left\{ 1 + n 2^{n+1} (e_3)^n/\delta + 2 n^{5/2} 2^{2n} (e_3)^{2n}/\delta^2 \right\}$$

It is important to note that the ratios $(e_3)^n/\delta$ and $(e_3)^{2n}/\delta^2$ are “dimensionless”, so the estimate (46) is scale invariant, in that the expression in brackets on the right hand side is unchanged by scalar multiplication on \mathbb{R}^n .

We next give an estimate for δ , the constant in (46) which is a lower bound on the absolute value of the determinant of \mathbf{U} , based on the geometry of the column vectors \vec{u}_k of \mathbf{U}^t . Recall that $|\mathbf{U}|$ equals the volume of the parallelepiped $P(\vec{y}_0, \vec{y}_1, \dots, \vec{y}_n)$ with edges $\{\vec{u}_1, \dots, \vec{u}_n\}$. The next definition is a natural geometric condition, which can be alternately formulated in terms of the angles formed by the vectors $\{\vec{u}_1, \dots, \vec{u}_n\}$, although for our purposes, the following is the preferable notion.

DEFINITION 10.5. *Let $\rho > 0$ and $1 \leq m \leq n$. A collection of vectors $\{\vec{y}_0, \dots, \vec{y}_m\} \subset \mathbb{R}^n$ is said to be ρ -robust if for each $0 \leq k < m$, the distance from the point \vec{y}_{k+1} to the affine subspace spanned by the vertices $\{\vec{y}_0, \dots, \vec{y}_k\}$ is at least ρ .*

The significance of this definition is seen from an elementary estimation, whose proof follows by induction and standard Euclidean geometry.

LEMMA 10.6. *Given a ρ -robust collection $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$, the parallelepiped $P(\vec{y}_0, \vec{y}_1, \dots, \vec{y}_n)$ has volume at least $\rho^{n-1} \cdot \|\vec{y}_1 - \vec{y}_0\|$. \square*

This volume estimate can be improved when the vertices are lattice points on an inscribed sphere:

LEMMA 10.7. *For $0 < e_1 < e_2$, there exists $V_2(e_1, e_2) > 0$ such that given $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$, and $0 < r \leq e_2$ satisfying:*

- (1) $e_1 \leq \|\vec{y}_k - \vec{y}_j\|$ for $0 \leq j \neq k \leq n$,
- (2) $\|\vec{y}_k\| = r$ for all $0 \leq k \leq n$,
- (3) $\{\vec{y}_0, \dots, \vec{y}_n\}$ is ρ -robust.

Then the parallelepiped $P(\vec{y}_0, \vec{y}_1, \dots, \vec{y}_n)$ has volume at least $V_2(e_1, e_2) \cdot \rho^{n-2}$.

Proof. First, note that the vectors $\{\vec{y}_0, \vec{y}_1, \vec{y}_2\} \subset \mathbb{R}^n$ cannot be collinear, as they lie on a sphere of radius $r \leq e_2$. Also, the vectors $\vec{u}_1 = \vec{y}_1 - \vec{y}_0$ and $\vec{u}_2 = \vec{y}_2 - \vec{y}_0$ have lengths greater than e_1 by (10.7.1), and thus define a non-degenerate parallelogram $P(\vec{y}_0, \vec{y}_1, \vec{y}_2)$. The minimum for the area over all such parallelograms must be positive, as these conditions define a compact set of such, all of which have positive area. Let $V_2(e_1, e_2) > 0$ denote this minimum.

Next, the vector \vec{y}_3 lies at distance at least ρ from the plane spanned by $\{\vec{y}_0, \vec{y}_1, \vec{y}_2\}$ by the ρ -robust assumption. As \vec{y}_0 lies on this plane, $\vec{u}_3 = \vec{y}_3 - \vec{y}_0$ must also lie distance at least ρ from it. Thus, the parallelepiped $P(\vec{y}_0, \vec{y}_1, \vec{y}_2, \vec{y}_3)$ with edges by $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ has volume bounded below by $V_2(e_1, e_2) \cdot \rho$.

Continuing by induction, one has that the parallelepiped $P(\vec{y}_0, \vec{y}_1, \dots, \vec{y}_k)$ with edges $\{\vec{u}_1, \dots, \vec{u}_k\}$ has volume bounded below by $V_2(e_1, e_2) \cdot \rho^{k-2}$ for all $2 < k \leq n$. \square

Lemma 10.7 hints at a fundamental difference between the study of Delaunay triangulations in dimension 2, and the theory for dimensions greater than two: the estimate for the volume of simplices in dimension two admits a uniform lower positive bound depending only on the constants $0 < e_1 < e_2$. While for higher dimensions, there is an additional restriction required to obtain such an estimate, the *robustness* of the vertices. The robustness can also be defined in terms of the interior angles of the simplex, in which case some form of this observation is almost certainly folklore.

We combine the above results to obtain the final form (47) of the desired estimate:

PROPOSITION 10.8. *Let $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$ be $\rho > 0$ robust, and admit an inscribed sphere with center $\omega(\vec{y}_0, \dots, \vec{y}_n)$ and radius $r(\vec{y}_0, \dots, \vec{y}_n)$. For $0 < e_1 < e_2$ set $\delta = V_2(e_1, e_2) \cdot \rho^{n-2}$, and let $\varepsilon > 0$, then suppose that*

- (1) $e_1 \leq \|\vec{y}_i - \vec{y}_j\|$ for all $0 \leq i \neq j \leq n$
- (2) $e_1/2 \leq r(\vec{y}_0, \dots, \vec{y}_n) \leq e_2$

$$(3) \quad \varepsilon \leq \delta/2^{n+1}n^{3/2}(e_2)^{n-1} \leq \frac{V_2(e_1, e_2) \cdot \rho^{n-2}}{2^{n+1}n^{3/2}(e_2)^{n-1}}$$

Let $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$ satisfy

$$(4) \quad \|\vec{y}_i - \vec{z}_i\| \leq \varepsilon \quad \text{for all } 0 \leq i \leq n,$$

then $\{\vec{z}_0, \dots, \vec{z}_n\}$ has an inscribed sphere with center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ so that for $e_3 = e_2 + \varepsilon$,

$$(47) \quad \|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\| < \varepsilon \cdot \left\{ 1 + n 2^{n+1} (e_3)^n / \delta + 2 n^{5/2} 2^{2n} (e_3)^{2n} / \delta^2 \right\}$$

11. INSCRIBED SPHERES VIA INEQUALITIES

There is an alternative approach to showing the existence of an inscribed sphere for points $\{\vec{z}_0, \dots, \vec{z}_n\}$, based on being given an “approximate solution” to the problem defined by a system of inequalities. This approach is advantageous when considering perturbations of a given triangulation, and we develop some key estimates which are used later.

Given vectors $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$, let \mathbf{V} denote the $n \times n$ matrix whose rows are the transposes of the vectors $\vec{v}_k = \vec{z}_{k-1} - \vec{z}_n$ for $1 \leq k \leq n$, and $\vec{\lambda}(\mathbf{V}) = (\|\vec{v}_1\|^2, \dots, \|\vec{v}_n\|^2)^t$. Assuming that \mathbf{V} is invertible, the first result gives an estimate on the distance between an approximate center for the points and the actual center.

PROPOSITION 11.1. *Suppose that we are given vectors $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$, $\omega \in \mathbb{R}^n$ and constants $0 < C_1 < r$ and $C_2 > 0$ such that*

- (1) $r - C_1 < \|\vec{z}_k - \omega\| < r + C_1$ for all $0 \leq k \leq n$,
- (2) $\|\mathbf{V}^{-1}\| \leq C_2$.

Then $\{\vec{z}_0, \dots, \vec{z}_n\}$ has an inscribed sphere with center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ such that

$$(48) \quad \|\omega - \omega(\vec{z}_0, \dots, \vec{z}_n)\| < 2\sqrt{n} \cdot r C_1 C_2$$

Proof. The existence of the inscribed sphere follows as before, given that \mathbf{V} is invertible. The center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ lies in the common intersection of the hyperplanes

$$\begin{aligned} L(\vec{z}_{k-1}, \vec{z}_n) &= \{\vec{x} \in \mathbb{R}^n \mid (\vec{z}_{k-1} - \vec{z}_n) \bullet (\vec{x} - (\vec{z}_{k-1} + \vec{z}_n)/2) = 0\} \\ &= \{\zeta + \vec{z}_n \mid \vec{v}_k \bullet \zeta = 1/2 \cdot \|\vec{v}_k\|^2\} \end{aligned}$$

where $\zeta = \vec{x} - \vec{z}_n$. Thus, the solution of the matrix equation (34),

$$(49) \quad \mathbf{V} \cdot \zeta = \frac{1}{2} \cdot \vec{\lambda}(\mathbf{V}) \quad , \quad \vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) = \frac{1}{2} \cdot \left\{ \mathbf{V}^{-1} \cdot \vec{\lambda}(\mathbf{V}) \right\} + \vec{z}_n$$

is the center for an inscribed sphere containing the points $\{\vec{z}_0, \dots, \vec{z}_n\}$. We must estimate $\|\omega'\|$ where $\omega' = \omega - \omega(\vec{z}_0, \dots, \vec{z}_n)$.

As $r - C_1 > 0$, the vector ω satisfies the inequalities

$$(50) \quad (r - C_1)^2 < \|\vec{z}_k - \omega\|^2 < (r + C_1)^2$$

Make the change of variables $\vec{v}_k = \vec{z}_{k-1} - \vec{z}_n$ and $\zeta = \omega - \vec{z}_n$, then expand to obtain

$$(51) \quad r^2 - 2rC_1 + C_1^2 < \|\vec{v}_k - \zeta\|^2 < r^2 + 2rC_1 + C_1^2$$

Note that $\vec{v}_{n+1} = \vec{z}_n - \vec{z}_n = \vec{0}$. Subtracting the inequalities (50) for $k = n + 1$ from those for $1 \leq k \leq n$ and expanding and canceling terms then yields

$$\begin{array}{llll} -4rC_1 & < & (\vec{v}_k - \zeta) \bullet (\vec{v}_k - \zeta) - (\vec{v}_{n+1} - \zeta) \bullet (\vec{v}_{n+1} - \zeta) & < & 4rC_1 \\ -4rC_1 & < & (\vec{v}_k \bullet \vec{v}_k - 2\vec{v}_k \bullet \zeta + \zeta \bullet \zeta) - \zeta \bullet \zeta & < & 4rC_1 \\ -4rC_1 & < & \vec{v}_k \bullet \vec{v}_k - 2\vec{v}_k \bullet \zeta & < & 4rC_1 \end{array}$$

Condition (11.1.1) and the above implies that $\zeta = \omega - \vec{z}_n$ is a solution of the matrix inequality

$$(52) \quad \mathbf{V} \cdot \zeta - \frac{1}{2} \vec{\lambda}(\mathbf{V}) \in B(0, 2\sqrt{n} \cdot r C_1)$$

The equation (49) implies that $\omega(\vec{z}_0, \dots, \vec{z}_n) - \vec{z}_n$ is a solution to the equation

$$(53) \quad \mathbf{V} \cdot \zeta - \frac{1}{2} \vec{\lambda}(\mathbf{V}) = \vec{0}$$

Thus, $\omega' = \omega - \omega(\vec{z}_0, \dots, \vec{z}_n)$ is a solution of the matrix inequality

$$(54) \quad \mathbf{V} \cdot \omega' \in B(0, 2\sqrt{n} \cdot r C_1)$$

We are given that $\|\mathbf{V}^{-1}\| \leq C_2$ hence we obtain the estimate (48). \square

The stability of the Delaunay triangulation associated to a net $\mathcal{M} \subset \mathbb{R}^n$ under perturbation of \mathcal{M} is equivalent to the stability of the inscribed spheres for the vertices of a simplex. The following result shows the existence of inscribed spheres based on estimates which are almost “stable under sufficiently small” perturbation.

PROPOSITION 11.2. *Let $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$ be $\rho > 0$ robust, and suppose there exists constants $0 < e_1 < e_2$ and $0 < C_1 < r < e_1$. Assume that there exists $\omega \in \mathbb{R}^n$ such that*

- (1) $e_1 < \|\vec{z}_i - \vec{z}_j\| < 2e_2$ for all $0 \leq i \neq j \leq n$
- (2) $r - C_1 < \|\vec{z}_k - \omega\| < r + C_1$ for all $0 \leq k \leq n$,

Then $\{\vec{z}_0, \dots, \vec{z}_n\}$ has an inscribed sphere with center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ so that for

$$(55) \quad \|\omega - \vec{\omega}(\vec{z}_0, \dots, \vec{z}_n)\| \leq C_1 \cdot n^{3/2} (2e_2)^{n-1} / \rho^{n-1}$$

Proof. Lemma 10.6 implies that the volume of the parallelepiped $P(\vec{z}_0, \dots, \vec{z}_n)$ with edges $\{\vec{v}_1, \dots, \vec{v}_n\}$ is bounded below by $e_1 \rho^{n-1}$, and hence $|\mathbf{V}| \geq e_1 \rho^{n-1}$. Thus by Corollary 10.3, we have

$$(56) \quad \|\mathbf{V}^{-1}\| \leq n(2e_2)^{n-1} / |\mathbf{V}| \leq n \cdot (2e_2)^{n-1} / e_1 \rho^{n-1}$$

Then (55) follows from estimate (48) of Proposition 11.1 and the hypotheses $r \leq e_1$. \square

Propositions 10.8 and 11.2 show the importance of the robustness condition in Definition 10.5 for estimating the stability of solutions for the equations (33). Our next result gives a stability estimate for the robustness of a vertices in a Delaunay triangulation.

PROPOSITION 11.3. *Let $1 \leq m \leq n$, and assume that $\{\vec{y}_0, \dots, \vec{y}_m\} \subset \mathbb{R}^n$ is ρ -robust. Let $\{\vec{z}_0, \dots, \vec{z}_m\} \subset \mathbb{R}^n$ be also given, along with the constants $0 < e_1 < e_2$ and $0 < \varepsilon < e_1/4$ such that*

- (1) $e_1 \leq \|\vec{y}_i - \vec{y}_j\| \leq 2e_2$ for all $0 \leq i \neq j \leq m$
- (2) $\|\vec{y}_i - \vec{z}_i\| \leq \varepsilon$ for all $0 \leq i \leq m$.

Then $\{\vec{z}_0, \dots, \vec{z}_m\}$ is ρ_m -robust, for $\rho_m = \rho_m(\rho, \varepsilon, e_1, e_2)$ as defined below. Moreover, $\rho_m(\rho, \varepsilon, e_1, e_2)$ is monotone increasing in e_2 and ρ , and monotone decreasing in e_1 and ε , and is scale-invariant. That is, for $s > 0$, $\rho_m(s \cdot \rho, s \cdot \varepsilon, s \cdot e_1, s \cdot e_2) = s \cdot \rho_m(\rho, \varepsilon, e_1, e_2)$.

Proof. Set $e'_1 = e_1 - 2\varepsilon$, $e'_2 = e_2 + \varepsilon$ and $e_4 = 4(e_2 + e_1)$. Then for all $0 \leq i \neq j \leq m$,

$$e_1/2 < e'_1 < \|\vec{z}_i - \vec{z}_j\| < 2e'_2 < e_4$$

For each $0 \leq k \leq m$, let $\text{Span}(\vec{y}_0, \dots, \vec{y}_k) \subset \mathbb{R}^n$ denote the affine subspace spanned by the vectors, and let $\xi_k \in \text{Span}(\vec{y}_0, \dots, \vec{y}_{k-1})$ be the point closest to \vec{y}_k . Then $\rho \leq \|\vec{y}_k - \xi_k\| \leq \|\vec{y}_k - \vec{y}_0\| \leq 2e_2$.

Similarly, let $\text{Span}(\vec{z}_0, \dots, \vec{z}_{k-1}) \subset \mathbb{R}^n$ denote the affine subspace spanned by the vectors, and $\zeta_k \in \text{Span}(\vec{z}_0, \dots, \vec{z}_{k-1})$ be the point closest to \vec{z}_k . Then $\|\vec{z}_k - \zeta_k\| \leq \|\vec{z}_k - \vec{z}_j\| < 2e'_2$ for $j \leq k-1$.

The triangle inequality yields a lower bound

$$(57) \quad \begin{aligned} d_{\mathbb{R}^m}(\vec{z}_k, \text{Span}(\vec{z}_0, \dots, \vec{z}_{k-1})) = \|\vec{z}_k - \zeta_k\| &\geq \|\vec{y}_k - \xi_k\| - \|\vec{z}_k - \vec{y}_k\| - \|\xi_k - \zeta_k\| \\ &\geq \rho - \varepsilon - \|\xi_k - \zeta_k\| \end{aligned}$$

Thus, it suffices to give an upper bound estimate for $\|\xi_k - \zeta_k\|$.

For the case $k = 1$, note that $\text{Span}(\vec{z}_0) = \{\vec{z}_0\}$ is just the single point, so $\xi_1 = \vec{y}_0$ and $\zeta_1 = \vec{z}_0$, and $\|\zeta_1 - \xi_1\| = \|\vec{z}_0 - \vec{y}_0\| \leq \varepsilon$, so in terms of the estimate (57) we have $d_{\mathbb{R}^m}(\vec{z}_1, \text{Span}(\vec{z}_0)) \geq \rho - 2\varepsilon$. Set $\delta_1 = 2$, then $\rho_1 = \rho - \varepsilon \cdot \delta_1$. If $m = 1$, then we are done.

For $k \geq 2$, the equations yielding an upper bound estimate on $\|\xi_k - \zeta_k\|$ are more delicate.

We are given that $\vec{y}_j, \vec{z}_j \in D_{\mathbb{R}^n}(\vec{y}_k, 2e_2 + \varepsilon)$ for each $0 \leq j \leq m$. Since the distance from \vec{y}_k to ξ_k is at most that from \vec{y}_k to \vec{y}_0 we also have $\xi_k \in D_{\mathbb{R}^n}(\vec{y}_k, 2e_2)$. The analogous estimate is true for ζ_k and since $\|\vec{y}_k - \vec{z}_k\| \leq \varepsilon$ we have that $\xi_k \in D_{\mathbb{R}^n}(\vec{y}_k, 2e_2 + \varepsilon)$. Thus, all of the points in consideration lie in the closed disk $D_{\mathbb{R}^n}(\vec{y}_k, e_4)$ with diameter e_4 . This compactness estimate is fundamental.

Let $\text{Span}_k(\vec{y}_0, \dots, \vec{y}_{k-1}) = \text{Span}(\vec{y}_0, \dots, \vec{y}_{k-1}) \cap D_{\mathbb{R}^n}(\vec{y}_k, 2e'_2)$. Note that we showed above that $\{\vec{y}_0, \dots, \vec{y}_{k-1}, \xi_1, \dots, \xi_k\} \subset \text{Span}_k(\vec{y}_0, \dots, \vec{y}_{k-1})$.

For the case $k = 2$, note that $\|\vec{y}_1 - \vec{y}_0\| \geq e_1$ and $\|\vec{z}_1 - \vec{z}_0\| \geq e'_1 > e_1/2$, and using that the disk $D_{\mathbb{R}^n}(\vec{y}_2, 2e'_2)$ has diameter at most e_4 , we have

$$(58) \quad \text{Span}_2(\vec{y}_0, \vec{y}_1) \subset \{\vec{y}_0 + t_1(\vec{y}_1 - \vec{y}_0) \mid -e_4/e_1 \leq t_1 \leq e_4/e_1\}$$

$$(59) \quad \text{Span}_2(\vec{z}_0, \vec{z}_1) \subset \{\vec{z}_0 + s_1(\vec{z}_1 - \vec{z}_0) \mid -e_4/e'_1 \leq s_1 \leq e_4/e'_1\}$$

LEMMA 11.4. *Given $\vec{z} \in \text{Span}_2(\vec{z}_0, \vec{z}_1)$, there exists $\vec{y} \in \text{Span}(\vec{y}_0, \vec{y}_1)$ so that*

$$(60) \quad \|\vec{z} - \vec{y}\| \leq \varepsilon \cdot (1 + 4e_4/e_1)$$

Proof. The point $\vec{z} \in \text{Span}_2(\vec{z}_0, \vec{z}_1)$ can be written as $\vec{z} = \vec{z}_0 + s_1 \cdot (\vec{z}_1 - \vec{z}_0) \in \text{Span}_2(\vec{z}_0, \vec{z}_1)$.

Then for $\vec{y} = \vec{y}_0 + s_1 \cdot (\vec{y}_1 - \vec{y}_0) \in \text{Span}_2(\vec{y}_0, \vec{y}_1)$ we have

$$\begin{aligned} \|\vec{z} - \vec{y}\| &= \|\{\vec{z}_0 + s_1 \cdot (\vec{z}_1 - \vec{z}_0)\} - \{\vec{y}_0 + s_1 \cdot (\vec{y}_1 - \vec{y}_0)\}\| \\ &\leq \|\vec{z}_0 - \vec{y}_0\| + |s_1| \|(\vec{z}_1 - \vec{z}_0) - (\vec{y}_1 - \vec{y}_0)\| \\ &\leq \varepsilon + |s_1|(\varepsilon + \varepsilon) \leq \varepsilon \cdot (1 + e_4/e'_1 \cdot 2\varepsilon) \leq \varepsilon \cdot (1 + 4e_4/e_1) \end{aligned}$$

Thus, every point of $\text{Span}_2(\vec{z}_0, \vec{z}_1)$ has distance at most $\varepsilon \cdot (1 + 4e_4/e_1)$ from a point of $\text{Span}(\vec{y}_0, \vec{y}_1)$. \square

Lemma 11.4 implies that $\|\xi_2 - \zeta_2\| \leq \varepsilon \cdot (1 + 4e_4/e_1)$, hence $\|\vec{z}_2 - \zeta_2\| \geq \rho_2$ by (57), where

$$(61) \quad \rho_2 = \rho - \varepsilon \cdot (2 + 4e_4/e_1) = \rho - \varepsilon \cdot \delta_2(\rho, e_1, e_2)$$

Note that $\delta_2(\rho, e_1, e_2) = (2 + 4e_4/e_1)$ depends only on the constants e_1 and e_2 , and as the ratio e_4/e_1 is scale invariant, thus ρ_2 is also scale invariant. If $m = 2$ then we are done.

Next, consider the case $k = 3$. The estimate ρ_3 in this case is obtained from (57) by subtracting from ρ a term which involves linear combinations of \vec{y}_2 with points of the line $\text{Span}(\vec{y}_0, \vec{y}_1)$, and the closer that \vec{y}_2 lies to this line, the larger the possible error, and likewise for $\text{Span}_3(\vec{z}_0, \vec{z}_1, \vec{z}_2)$.

As seen before for $k = 2$, the strategy is to estimate the parameters used to describe the planar region $\text{Span}_3(\vec{y}_0, \vec{y}_1, \vec{y}_2)$ as in (58), and similarly for $\text{Span}_3(\vec{z}_0, \vec{z}_1, \vec{z}_2)$ as in (59).

Recall that $\xi_2 \in \text{Span}(\vec{y}_0, \vec{y}_1)$ is the point on the line closest to \vec{y}_2 , and $\rho \leq \|\vec{y}_2 - \xi_2\| \leq 2e_2 < e_4$.

Likewise, the point $\zeta_2 \in \text{Span}(\vec{z}_0, \vec{z}_1)$ closest to \vec{z}_2 satisfies $\rho_2 \leq \|\vec{z}_2 - \zeta_2\| \leq 2e'_2 < e_4$.

Now let $\xi'_2 \in \text{Span}(\vec{y}_0, \vec{y}_1)$ be the point closest to ζ_2 . Then $\|\vec{y}_2 - \xi'_2\| \geq \|\vec{y}_2 - \xi_2\| \geq \rho > \rho_2$.

Furthermore, from the case $k = 2$, we have that $\|\xi'_2 - \zeta_2\| \leq \varepsilon \cdot \delta_2(\rho, e_1, e_2)$.

The key idea is to bound the space $\text{Span}_3(\vec{y}_0, \vec{y}_1, \vec{y}_2)$ using linear combinations with $(\vec{y}_2 - \xi'_2)$ and parameter bounds invoking ρ and ρ_2 :

$$\begin{aligned} \text{Span}_3(\vec{y}_0, \vec{y}_1, \vec{y}_2) &\subset \{ \vec{y}_0 + t_1(\vec{y}_1 - \vec{y}_0) + t_2(\vec{y}_2 - \xi'_2) \mid -e_4/e_1 \leq t_1 \leq e_4/e_1, -e_4/\rho \leq t_2 \leq e_4/\rho \} \\ \text{Span}_3(\vec{z}_0, \vec{z}_1, \vec{z}_2) &\subset \{ \vec{z}_0 + s_1(\vec{z}_1 - \vec{z}_0) + s_2(\vec{z}_2 - \zeta_2) \mid -e_4/e'_1 \leq s_1 \leq e_4/e'_1, -e_4/\rho_2 \leq s_2 \leq e_4/\rho_2 \} \end{aligned}$$

As in the proof of Lemma 11.4, every point of $\text{Span}_3(\vec{z}_0, \vec{z}_1, \vec{z}_2)$ thus lies a distance at most

$$\varepsilon \cdot \{1 + 2e_4/e_1 \cdot (1 + 1) + 2e_4/\rho_2 \cdot (1 + \delta_2)\}$$

from a point of $\text{Span}_3(\vec{y}_0, \vec{y}_1, \vec{y}_2)$, and in particular this estimate holds for $\|\xi_3 - \zeta_3\|$. Set

$$(62) \quad \delta_3 = \delta_3(\rho, e_1, e_2) = 2 + 4e_4/e_1 + (1 + \delta_2) \cdot 2e_4/\rho_2$$

Note that the ratio e_4/ρ_2 is scale-invariant, as is δ_2 , and thus δ_3 is scale-invariant.

Then for $\rho_3 = \rho - \varepsilon \cdot \delta_3$ by (57) we have $\|\vec{z}_3 - \zeta_3\| \geq \rho_3$.

Continuing in the way, given ρ_k and δ_k for $2 \leq k < m$, define inductively

$$(63) \quad \delta_{k+1} = 1 + \{1 + 2 \cdot 2e_4/e_1 + (1 + \delta_2) \cdot 2e_4/\rho_2 + \cdots + (1 + \delta_k) \cdot 2e_4/\rho_k\}$$

$$(64) \quad \rho_{k+1} = \rho - \varepsilon \cdot \delta_{k+1}$$

Then we have $\|\vec{z}_{k+1} - \zeta_{k+1}\| \geq \rho_{k+1}$. Continuing until $k + 1 = m$, we obtain

$$(65) \quad \delta_m(\rho, e_1, e_2) = 2 + 4e_4/e_1 + 2 \cdot \sum_{k=2}^{m-1} \frac{(1 + \delta_k)e_4}{\rho_k}$$

$$(66) \quad \rho_m(\rho, \varepsilon, e_1, e_2) = \rho - \varepsilon \cdot \delta_m(\rho, e_1, e_2)$$

for which $d_{\mathbb{R}^m}(\vec{z}_m, \text{Span}(\vec{z}_0, \dots, \vec{z}_{m-1})) = \|\vec{z}_m - \zeta_m\| \geq \rho_m(\rho, \varepsilon, e_1, e_2)$.

Observe that by the inductive definition (64), the values $\rho > \rho_1 > \cdots > \rho_m$ are monotone decreasing. Furthermore, by an inductive argument, for each $1 \leq k < m$ the value of ρ_k is a monotone increasing function of e_2 and ρ , and monotone decreasing for e_1 , and thus each term $(1 + \delta_k)e_4/\rho_k$ in the sum (65) is also monotone increasing, hence the same holds for $\rho_m(\rho, \varepsilon, e_1, e_2)$. Also note that each additional term $(1 + \delta_k) \cdot 2e_4/\rho_k$ in (63) is scale-invariant, so the sum (66) is scale-invariant. \square

PART IV - MICRO-LOCAL RIEMANNIAN GEOMETRY

In Part IV, we discuss the “micro-local Riemannian geometry” of a matchbox manifold \mathfrak{M} . The goal is to extend several key concepts of Part III to the leaves of \mathfrak{M} . This requires a sequence of estimates, the result of which is that the leafwise disks of a fixed radius can be assumed to be “ ε -approximately Euclidean”, and vary in the transverse direction by a controlled amount. While the development in Section 12 is a bit tedious, the material in Section 13 is without doubt difficult to get through. The reader can omit both these sections on first reading, but their inclusion is justified by the need to provide full details of the construction.

12. MICRO-LOCAL FOLIATION GEOMETRY

Recall that for each $1 \leq i \leq \nu$ the coordinate chart $\varphi_i: \overline{U}_i \rightarrow [-1, 1]^n \times \mathfrak{T}_i$ and for $x \in \overline{U}_i$, $\mathcal{P}_i(x)$ is the plaque for the chart φ_i containing x . Moreover, we define $\lambda_i: \overline{U}_i \rightarrow [-1, 1]^n$ by setting $\varphi_i(x) = (\lambda_i(x), w_x) \in [-1, 1]^n \times \mathfrak{T}_i$. The map λ_i defines the smooth structure on each plaque $\mathcal{P}_i(x)$.

Also, recall that $\lambda_{\mathcal{F}} > 0$ was chosen in Lemma 2.5 so that for all $x \in \mathfrak{M}$, the closed leafwise disk $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}})$ is strongly convex, and the constant $\delta_{\mathcal{U}}^{\mathcal{F}}$ is the fixed radius of the plaques in the foliation covering, and satisfies $\delta_{\mathcal{U}}^{\mathcal{F}} < \lambda_{\mathcal{F}}/4$.

For $x \in \overline{U}_i$ define the transversal section, for $X \subset \mathfrak{T}_i$

$$(67) \quad \mathfrak{S}(x, i, X) \equiv \varphi_i^{-1}(\lambda_i(x), X) ; \mathfrak{S}(x, i) \equiv \varphi_i^{-1}(\lambda_i(x), \mathfrak{T}_i) = \lambda_i^{-1} \circ \lambda_i(x)$$

As a special case, for $r \geq 0$, define the compact “disk section”

$$(68) \quad \mathfrak{S}(x, i, r) \equiv \varphi_i^{-1}(\lambda_i(x), D_{\mathfrak{X}}(w_x, r) \cap \mathfrak{T}_i) \subset \overline{U}_i$$

The local coordinate charts $\varphi_i: \overline{U}_i \rightarrow [-1, 1]^n \times \mathfrak{T}_i$ are used to define a local “vertical translation” between plaques, which will be fundamental in the following. For $x' \in \mathfrak{S}(x, i)$, define

$$(69) \quad \phi_i(x, x'): \mathcal{P}_i(x) \rightarrow \mathcal{P}_i(x'), \quad \xi' = \phi(x, x')(\xi) = \mathfrak{S}(\xi, i) \cap \mathcal{P}_i(x')$$

When expressed in coordinates, the map ϕ_i becomes $\varphi_i \circ \phi_i(x, x') \varphi_i^{-1}(\lambda_i(x), w_x) = (\lambda_i(x), w_{x'})$, which is just the constant map in the first coordinate. Thus $\phi_i(x', x'') \circ \phi_i(x, x') = \phi_i(x, x'')$, and the maps $\phi_i(x, x')$ are homeomorphisms which depend continuously on $x' \in \mathfrak{T}_i$ in the C^0 -topology.

The construction of a stable nice transversal for \mathfrak{M} is based on the construction of nice Delaunay triangulations of all leaves, which have strong invariance properties with respect to the maps $\phi_i(x, x')$ for all coordinate charts on \mathfrak{M} . The construction we give in the Section 14 requires very fine control on the metric distortions of the transverse translations $\phi_i(x, x')$.

We introduce estimates on the *leafwise metric* distortions of the maps $\phi_i(x, x')$. First, compare the Riemannian distance functions induced on differing plaques in the same chart \overline{U}_i by defining:

$$(70) \quad \begin{aligned} \text{var}(i, r) &= \max \{ |d_{\mathcal{F}}(x, y) - d_{\mathcal{F}}(x', y')| \mid x \in \overline{U}_i, x' \in \mathfrak{S}(x, i, r), y \in \mathcal{P}_i(x), y' = \phi(x, x')(y) \} \\ &= \max \{ \{ |d_{\mathcal{F}}(y, z) - d_{\mathcal{F}}(\phi_i(x, x')(y), \phi_i(x, x')(z))| \} \mid y, z \in \mathcal{P}_i(x), x' \in \mathfrak{S}(x, i, r) \} \end{aligned}$$

Note that $\text{var}(i, r)$ depends continuously on r , that $\text{var}(i, 0) = 0$, and $\text{var}(i, r) \leq 2\epsilon_{\mathcal{U}}^{\mathcal{F}}$ as $\mathcal{P}_i(x)$ is a disk in L_x of radius $\delta_{\mathcal{U}}^{\mathcal{F}}$.

There is another measure of the metric distortion between plaques, this time in terms of the variation due to differing coordinate systems. For $z \in \overline{U}_i \cap \overline{U}_j$ and $i \neq j$, we obtain two transversals $\mathfrak{S}(z, i, r)$ and $\mathfrak{S}(z, j, r)$ through z in \mathfrak{M} . Define the *divergence* between these two transversals by

$$(71) \quad \text{div}(z, i, j, r) = \max \{ d_{\mathcal{F}}(x', y') \mid x' \in \mathfrak{S}(z, i, r), y' \in \mathfrak{S}(z, j, r), \mathcal{P}_i(x') \cap \mathcal{P}_j(y') \neq \emptyset \}$$

The assumption $\delta_{\mathcal{U}}^{\mathcal{F}} < \lambda_{\mathcal{F}}/4$ implies that $\text{div}(z, i, j, r) < \lambda_{\mathcal{F}}$. Note that $\text{div}(z, i, i, r) = 0$ and that $\text{div}(z, i, j, 0) = 0$. Define

$$(72) \quad \text{div}(z, r) = \max \{ \text{div}(z, i, j, r) \mid z \in \overline{U}_i \cap \overline{U}_j \}$$

The condition $\mathcal{P}_i(x') \cap \mathcal{P}_j(y') \neq \emptyset$ is closed in x', y' , and hence $\text{div}(z, r)$ is an upper semi-continuous function of both z and r . In terms of the transverse translation maps ϕ_i , for $\varepsilon = \text{div}(z, r)$, the condition (72) implies that the compositions $\phi_i(x', z) \circ \phi_j(z, y')$ are ε -close to the identity on the appropriate domains.

Finally, we consider the distortion in sufficiently small leafwise disks for geodesic coordinates.

Let $\hat{e} \equiv \{\vec{e}_1, \dots, \vec{e}_n\}$ denote the standard orthonormal basis of \mathbb{R}^n . A point $\vec{x} \in \mathbb{R}^n$ is then written in coordinates as $\vec{a} = (a_1, \dots, a_n)$, where $\vec{x} = \hat{e} \cdot \vec{a} = a_1 \vec{e}_1 + \dots + a_n \vec{e}_n$. Recall that the closed ball of radius λ about the origin in the standard metric is denoted by $D_{\mathbb{R}^n}(\lambda)$, or just $D(\lambda)$ when there is no chance of confusion, and that $\|\cdot\| = \|\cdot\|_{\mathbb{R}^n}$ denotes the standard norm.

For $x \in \mathfrak{M}$, and coordinate system φ_i with $x \in U_i$ then \hat{e} defines a framing \hat{e}_w of $T_0(-1, 1)^n \times \{w\}$ for each $w \in \mathfrak{T}_i$ hence defines a framing \hat{e}_x for $T_x \mathcal{F}$ at each $x \in U_i$. Note that \hat{e} need not be orthonormal for the leafwise Riemannian metric. The linear isomorphism $T_x \mathcal{F} \cong \mathbb{R}^n$ induced by this framing is just a formal expression of the usual map induced by coordinates.

The leafwise Riemannian metric on $T\mathcal{F}$ induces on each plaque $\mathcal{P}_i(w)$ of U_i a family of inner products on the tangent spaces to $(-1, 1)^n \times \{w\}$, whose matrix in terms of the framing \hat{e}_x at $x \in \mathcal{P}_i(w)$ is denoted by $g_{jk}(x)$. By Theorem 2.3, the tensor $g_{jk}(x)$ varies continuously in $w \in \mathfrak{T}_i$ for the C^∞ -topology on functions on $\mathcal{P}_i(w)$.

Given an orthonormal frame $\hat{u} = \{\vec{u}_1, \dots, \vec{u}_n\} \subset T_x \mathcal{F}$ for the leafwise Riemannian metric, adopt the “matrix notation” $\hat{u} \cdot \vec{a} \equiv a_1 \vec{u}_1 + \dots + a_n \vec{u}_n \in T_x \mathcal{F}$. Via the coordinate isomorphism $T_x \mathcal{F} \cong \mathbb{R}^n$, the

vectors \vec{u}_k form an orthonormal set for the inner product $g_{jk}(x)$, and in this sense, $\hat{u} \cdot \vec{a}$ is precisely a matrix product. Define a linear isomorphism

$$(73) \quad F_{\hat{u}}: \mathbb{R}^n \rightarrow T_x \mathcal{F} \cong \mathbb{R}^n, \quad F_{\hat{u}}(a_1, \dots, a_n) = \hat{u} \cdot \vec{a}$$

Let $\|\cdot\|_{\hat{u}}$ denote the norm on $T_x \mathcal{F} \cong \mathbb{R}^n$ induced by the inner product $g_{ij}(x)$, then $F_{\hat{u}}$ is a linear isometry between $\{\mathbb{R}^n, \|\cdot\|\}$ and $\{\mathbb{R}^n, \|\cdot\|_{\hat{u}}\}$.

Recall that $\exp_x^{\mathcal{F}}: T_x \mathcal{F} \rightarrow L_x$ is the leafwise geodesic map at x . Given an orthonormal framing \hat{u} of $T_x \mathcal{F}$ and $0 < \lambda \leq \lambda_{\mathcal{F}}$, the *leafwise geodesic coordinates* at x are defined by

$$(74) \quad \varphi_{x,\hat{u}}^g: D_{\mathbb{R}^n}(\lambda) \rightarrow D_{\mathcal{F}}(x, \lambda) \subset L_x, \quad \varphi_{x,\hat{u}}^g(\vec{a}) = \exp_x^{\mathcal{F}}(\hat{u} \cdot \vec{a})$$

Assume that $D_{\mathcal{F}}(x, \lambda) \subset U_i$, and let $\vec{x} = \lambda_i(x) \in \mathbb{R}^n$. Then we have a second coordinate system on the neighborhood $D_{\mathcal{F}}(x, \lambda)$ of x , which is also “adapted” to the leafwise Riemannian metric on the disk $D_{\mathcal{F}}(x, \lambda)$. Define $T_{\vec{x}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T_{\vec{x}}(\vec{y}) = \vec{y} + \vec{x}$. Then compose $T_{\vec{x}}$ with the framing map $F_{\hat{u}}$ to obtain

$$(75) \quad \varphi_{x,\hat{u}}^i \equiv \varphi_i^{-1}(T_{\vec{x}} \circ F_{\hat{u}}, w_x): D(\lambda) \rightarrow \mathcal{P}_i(x), \quad \varphi_{x,\hat{u}}^i(\vec{y}) = \varphi_i(\vec{x} + \hat{u} \cdot \vec{y}, w_x)$$

which is just the geodesic coordinates for the flat metric on $\mathcal{P}_i(x)$ associated to the framing \hat{u} .

We compare the *affine geometries* defined by these two sets of coordinates, using the coordinate system φ_i to transfer the task to a problem of differential equations on \mathbb{R}^n .

Let $\tilde{D}_i(\vec{x}, \lambda) = \varphi_i(D_{\mathcal{F}}(x, \lambda)) \subset (-1, 1)^n \times \{w_x\}$ be the image of the geodesic disk $\varphi_{x,\hat{u}}^g(D(\lambda))$.

Let \tilde{d} denote the distance function on $\tilde{D}_i(\vec{x}, \lambda)$ defined by the leafwise metric $d_{\mathcal{F}}$ via the coordinates φ_i . That is, for $\vec{y}, \vec{z} \in \tilde{D}_i(\vec{x}, \lambda)$, $\tilde{d}(\vec{y}, \vec{z}) = d_{\mathcal{F}}(\varphi_i^{-1}(\vec{y}, w_x), \varphi_i^{-1}(\vec{z}, w_x))$.

Let \tilde{g} denote the metric tensor on $\tilde{D}_i(\vec{x}, \lambda)$ in the coordinates φ_i . Note that the image under φ_i of a geodesic segment for g is a geodesic segment for \tilde{g} , and as $D_{\mathcal{F}}(x, \lambda)$ is strongly convex for $\lambda \leq \lambda_{\mathcal{F}}$, the same holds for the region $\tilde{D}_i(\vec{x}, \lambda)$ with the metric \tilde{g} .

Let $\widetilde{\exp}_{\vec{x}}$ denote the geodesic map associated to \tilde{g} , centered at $\vec{x} = \lambda_i(x)$. Then for the orthonormal basis \hat{u} of \mathbb{R}^n we set

$$(76) \quad \widetilde{\exp}_{\vec{x},\hat{u}}: D(\lambda) \rightarrow \tilde{D}_i(\vec{x}, \lambda), \quad \widetilde{\exp}_{\vec{x},\hat{u}}(\vec{a}) = \widetilde{\exp}_{\vec{x}}(\hat{u} \cdot \vec{a})$$

Recall that we also have a linear map $T_{\vec{x}} \circ F_{\hat{u}}$, which is a linear isometry between $\{\mathbb{R}^n, \|\cdot\|\}$ and $\{\mathbb{R}^n, \|\cdot\|_{\hat{u}}\}$, and satisfies $T_{\vec{x}} \circ F_{\hat{u}}(\vec{0}) = \vec{x} = \widetilde{\exp}_{\vec{x},\hat{u}}(\vec{0})$. Let $g^{\hat{u}} = (T_{\vec{x}} \circ F_{\hat{u}})^*(\tilde{g})$ denote the metric \tilde{g} near \vec{x} pulled back to $D(\lambda)$ via the isometry $T_{\vec{x}} \circ F_{\hat{u}}$. Then $g_{jk}^{\hat{u}}(\vec{a}) = \delta_{jk}$ for $\vec{a} = \vec{0}$ by definition of \hat{u} .

DEFINITION 12.1. *Let $x \in \mathfrak{M}$ and $0 < \lambda \leq \lambda_{\mathcal{F}}/2$. Assume that $D_{\mathcal{F}}(x, \lambda) \subset \mathcal{P}_i(x)$, and let \hat{u} be an orthonormal frame for $T_x \mathcal{F}$. For $\varepsilon > 0$, we say that $\varphi_{x,\hat{u}}^g: D(\lambda) \rightarrow D_{\mathcal{F}}(x, \lambda)$ is ε -approximately Euclidean if the following hold (in the coordinate system φ_i):*

- (1) For all $\vec{a} \in D(\lambda)$,
- (77)
$$\|g_{jk}^{\hat{u}}(\vec{a}) - \delta_{jk}\| \leq \varepsilon/n^2$$
- (2) For all $\vec{a} \in D(\lambda)$,
- (78)
$$\tilde{d}(\widetilde{\exp}_{\vec{x},\hat{u}}(\vec{a}), T_{\vec{x}} \circ F_{\hat{u}}(\vec{a})) \leq \varepsilon \cdot \|\vec{a}\|$$
- (3) For a geodesic $\tilde{\sigma}: [0, 1] \rightarrow \tilde{D}_i(\vec{x}, \lambda)$ in the \tilde{d} metric, with $\tilde{\sigma}(0) = \vec{y}_0$ and $\tilde{\sigma}(1) = \vec{y}_1$, set $\tilde{\tau}(t) = t \cdot (\vec{y}_1 - \vec{y}_0) + \vec{y}_0$, then
- (79)
$$\tilde{d}(\tilde{\sigma}(t), \tilde{\tau}(t)) \leq \varepsilon \cdot \tilde{d}(\vec{y}_0, \vec{y}_1), \text{ for all } 0 \leq t \leq 1$$
- (4) For $s \leq \lambda$, the Riemannian volume of leafwise disks satisfies
- (80)
$$|\text{Vol}(D(s)) - \text{Vol}_{\tilde{d}}(D_{\tilde{g}}(\vec{x}, s))| \leq \varepsilon \cdot s^n$$

where Vol denotes the Euclidean volume and $\text{Vol}_{\tilde{g}}$ the volume form for the metric \tilde{g} . More generally, given an open set $U \subset D(s)$, for $s \leq \lambda$, we require that

$$(81) \quad |\text{Vol}(U) - \text{Vol}_{\tilde{g}}(\widetilde{\text{exp}}_{\tilde{x}, \tilde{u}}(U))| \leq \varepsilon \cdot s^n$$

Conditions (12.1.1) and (12.1.4) concern the continuity of the metric tensor \tilde{g} , while conditions (12.1.2-3) concern the behavior of geodesics for the metric \tilde{g} , so also require control on the first and second order derivatives of \tilde{g} . The condition (12.1.3) is simply that the geodesics for the metric \tilde{g} and the flat metric defined by \tilde{u} “stay close”. The conditions (12.1.1-5) are closely related, but are stated separately in the form they will be cited later in the text. We mention one immediate implication of condition (12.1.1).

LEMMA 12.2. *Assume that $\varphi_{x, \tilde{u}}^g: D(\lambda) \rightarrow D_{\mathcal{F}}(x, \lambda)$ is ε -approximately Euclidean. Then*

$$(82) \quad |\tilde{d}(\tilde{z}, \tilde{y}) - \|\tilde{z} - \tilde{y}\|_{\tilde{u}}| \leq \varepsilon \cdot \|\tilde{z} - \tilde{y}\|_{\tilde{u}} \quad \text{for all } \tilde{y}, \tilde{z} \in \tilde{D}_i(\tilde{x}, \lambda)$$

Thus, conditions (12.1.1) and (12.1.2) yield, for all $\tilde{a} \in D(\lambda)$,

$$(83) \quad \|\widetilde{\text{exp}}_{\tilde{x}, \tilde{u}}(\tilde{a}) - T_{\tilde{x}} \circ F_{\tilde{u}}(\tilde{a})\|_{\tilde{u}} \leq 2\varepsilon\lambda$$

Proof. Condition (12.1.1) and the estimate (31) imply the bound $\|g^{\tilde{u}} - \delta\|_{\tilde{u}} \leq \varepsilon$ on the matrix norm, from which (82) follows. Condition (83) then follows, as $\|\tilde{a}\| \leq \lambda$. \square

PROPOSITION 12.3. *Let $\varepsilon > 0$. Then there exists $\lambda_\varepsilon > 0$, so that for all $x \in \mathfrak{M}$ with $D_{\mathcal{F}}(x, \lambda_\varepsilon) \subset U_i$ and frame \tilde{u} of $T_x \mathcal{F}$, the chart $\varphi_{x, \tilde{u}}^g: D(\lambda_\varepsilon) \rightarrow L_x$ is ε -approximately Euclidean.*

Proof. The claim is that $\widetilde{\text{exp}}_{x, \tilde{u}}$ is well-approximated by the affine map $T_{\tilde{x}} \circ F_{\tilde{u}}$ for λ sufficiently small. This follows from standard facts about the geodesic charts for smooth metrics. (For example, see [22, Chapter 5].) The only novelty is that we use the continuity of the Riemannian metric and its derivatives as functions of $x \in \mathfrak{M}$ to obtain uniform estimates. We briefly sketch the arguments.

Let $\tilde{h} = \tilde{h}_{jk}(\xi)$ denote the Riemannian tensor on $D(\lambda)$ induced from \tilde{g} by the geodesic map $\widetilde{\text{exp}}_{x, \tilde{u}}$. Note that geodesic coordinates have the property that $\tilde{h}_{jk}(\vec{0}) = \delta_{jk}$, the Dirac δ -function. Moreover, the Riemannian Christoffel symbols $\tilde{\Gamma}_{jk}^\ell(\xi)$ of the metric \tilde{h} also vanish at the origin.

The tensor $\tilde{\Gamma}_{jk}^\ell(\xi)$ is $C^{\ell-1}$ -continuous as a function of the metric tensor in the C^ℓ topology, for $\ell \geq 1$, so the first derivatives of $\tilde{\Gamma}_{ij}^k(\xi)$ vary continuously with the metric in the C^2 topology, hence its curvature tensor $\tilde{R}(\xi)$ varies continuously in the C^2 topology as well. Thus, by choosing $\lambda > 0$ sufficiently small, we can assume the quantities $\|\tilde{h}_{jk}(\xi) - \delta_{jk}\|$ and $|\tilde{\Gamma}_{jk}^\ell(\xi)|$ are arbitrarily small on the disk $D(\lambda)$, and moreover its curvature tensor $|\tilde{R}(\xi)|$ is uniformly bounded.

Standard results of Riemannian geometry show that the second derivatives of the geodesic map $\widetilde{\text{exp}}_{x, \tilde{u}}$ at the origin are bounded by the norms of the Christoffel symbols $\tilde{\Gamma}_{jk}^\ell(\xi)$, their derivatives and the curvature terms $\tilde{R}(\xi)$. (For example, see [22, Chapter 5, Remark 2.11].) Thus, given $\varepsilon' > 0$, there exists $\lambda_{x, \varepsilon'} > 0$ such that $\widetilde{\text{exp}}_{x, \tilde{u}}$ is ε' -close to its linear approximation $T_{\tilde{x}} \circ F_{\tilde{u}}$ in the Euclidean norm on \mathbb{R}^n . This yields (78).

The condition (79) also follows, given that the local expressions of the Christoffel symbols $\tilde{\Gamma}_{jk}^\ell$ are sufficiently small on $D(\lambda)$ and the quantities $|\tilde{\Gamma}_{jk}^\ell|$ are uniformly bounded. Conditions (12.1.1) and (12.1.4-5) follow from the continuity of the metric tensor \tilde{g} , as noted above.

DEFINITION 12.4. *For each $\varepsilon > 0$, choose $\lambda_\varepsilon > 0$ so that the conditions of Definition 12.1 holds for all $x \in \mathfrak{M}$ and any choice of orthonormal frame.*

Thus, for each $\varepsilon > 0$, there exists an $\lambda_\varepsilon > 0$, so that the conditions of Definition 12.1 holds for all $x \in \mathfrak{M}$ and any choice of orthonormal frame.

The only subtlety here is that the error estimates (78) and (79) are in terms of the leafwise distance function $d_{\mathcal{F}}$, while the error ε' above is in terms of the Euclidean norm $\|\cdot\|$ on $D(\lambda)$. Introduce the constant

$$\|d_{\mathcal{F}}\| = \max \left\{ \frac{d_{\mathcal{F}}(\varphi_{x,\hat{u}}^g(\vec{b}), \varphi_{x,\hat{u}}^g(\vec{a}))}{\|\vec{b} - \vec{a}\|}, \frac{\|\vec{b} - \vec{a}\|}{d_{\mathcal{F}}(\varphi_{x,\hat{u}}^g(\vec{b}), \varphi_{x,\hat{u}}^g(\vec{a}))} \mid x \in \mathfrak{M}, \hat{u}, \vec{a} \neq \vec{b} \in D(\lambda_{\mathcal{F}}) \right\}$$

Given ε , let $\varepsilon' = \varepsilon/\|d_{\mathcal{F}}\|$ and choose λ_{ε} as above. By the compactness of \mathfrak{M} and the continuity of the metric properties used, given $\varepsilon > 0$ there exists an $\lambda_{\varepsilon} > 0$, so that for all $0 < \lambda$, $x \in \mathfrak{M}$ and $1 \leq i \leq \nu$ with $D_{\mathcal{F}}(x, \lambda) \subset \mathcal{P}_i(x)$, the estimates (78 – 80) of Definition 12.1 are satisfied. \square

If the leaves of \mathcal{F} are isometric to Euclidean space \mathbb{R}^n , such as in the case where \mathcal{F} is defined by a free action of \mathbb{R}^n , then λ_{ε} may be chosen arbitrarily large. Otherwise, if the leaves of \mathcal{F} have large sectional curvatures and ε is small, then λ_{ε} may be quite small. One consequence of $\lambda_{\varepsilon} \ll \lambda_{\mathcal{F}}$ is that it means the points in the leafwise nets constructed in section 14 will be very closely spaced.

13. SETTING THE CONSTANTS

We next fix various constants which appear later in calculations, and then draw some implications of these choices. First, we set

$$(84) \quad C_n = \frac{10^n!}{1!(10^n - 1)!} + \frac{10^n!}{2!(10^n - 2)!} + \cdots + \frac{10^n!}{n!(10^n - n)!} + \frac{10^n!}{(n+1)!(10^n - n - 1)!}$$

Given a finite subset $\Omega \subset D_{\mathcal{F}}(\xi, \lambda_{\mathcal{F}}^*)$ with cardinality bounded above by 10^n , then C_n is an upper bound for the number of distinct subsets of Ω consisting of at most $(n+1)$ -distinct points. In particular, C_n is an upper bound on the number of distinct n -simplices, defined by $(n+1)$ -vertices in Ω . Thus, C_n is an upper bound on the number of inscribed spheres for the set Ω .

Now introduce four “dimensionless” constants. The purpose of these choices is briefly indicated, and their precise roles will be apparent later. The constants are defined now, as it is fundamental that these can be chosen independent of later choices. In particular, the constants are scale-invariant, and are multiplied by the scale $\lambda_{\mathcal{F}}^*$ defined in (92) below in applications.

The width of the annular regions appearing in Lemma 14.2 is defined by

$$(85) \quad \varepsilon_1 = 1/(C_n \cdot 1000n \cdot 100^n)$$

The thickness of the rectangular regions appearing in the robustness condition (130) is defined by

$$(86) \quad \varepsilon_2 = 1/(C_n \cdot 2000 \cdot 2^n)$$

The constant ε_3 first appears in the statement and proof of Proposition 15.2, and is the basic estimate of the translation distance of the centers of inscribed spheres for perturbed vertices. We use repeatedly that $\varepsilon_3 < \varepsilon_1/4$ and so set

$$(87) \quad \varepsilon_3 = \varepsilon_1/10$$

The constant ε_4 is the error of the affine approximation introduced in Proposition 13.4. The value of this constant determines the recursive decrease in the robustness estimates in Propositions 11.3, 13.7 and 15.2. Proposition 11.3 gives a recursive definition for the functions $\rho_m(\rho, \varepsilon, e_1, e_2)$ for $1 \leq m \leq n$. As noted there, the function $\rho_m(\rho, \varepsilon, e_1, e_2)$ is monotone increasing in e_2 and ρ , and monotone decreasing in e_1 and ε , and satisfies $\rho_m(s \cdot \rho, s \cdot \varepsilon, s \cdot e_1, s \cdot e_2) = s \cdot \rho_m(\rho, \varepsilon, e_1, e_2)$ for $s > 0$. Moreover, for all $1 \leq m \leq n$, $\rho_m(\rho, 0, e_1, e_2) = \rho$.

For the normalized values $e_1 = 1$, $e_2 = 2$, $e_4 = 4(e_2 + e_1) = 12$, and $\rho = \rho_0$, define functions $\rho_m(\rho_0, \varepsilon)$ recursively by

$$\rho_0(\rho_0, \varepsilon) = \rho_0, \quad \delta_1 = 2, \quad \rho_1(\rho_0, \varepsilon) = \rho_0 - 2\varepsilon, \quad \delta_2 = 50, \quad \rho_2(\rho_0, \varepsilon) = \rho_0 - 50\varepsilon$$

and for $1 < m \leq n$, by

$$(88) \quad \rho_m(\rho_0, \varepsilon) = \rho_0 - \varepsilon \cdot \delta_m \quad , \quad \delta_m = 50 + 24 \cdot \sum_{k=2}^{m-1} \frac{(1 + \delta_k)}{\rho_k(\varepsilon)}$$

Note that each $\rho_m(\varepsilon)$ is a polynomial function of ε , so is continuous in ε . Also, for fixed (ρ_0, ε) , the sequence of functions are monotone decreasing in m :

$$\rho_0 = \rho_0(\rho_0, \varepsilon) > \rho_1(\rho_0, \varepsilon) > \rho_2(\rho_0, \varepsilon) > \cdots > \rho_n(\rho_0, \varepsilon) > 0$$

For each $0 \leq k \leq n$, set $\hat{\rho}_k = (18 - 2k/3n) \cdot \varepsilon_2$ and $\hat{\rho}'_k = (18 - (2k+1)/3n) \cdot \varepsilon_2$. Then we have

$$(89) \quad 18\varepsilon_2 = \hat{\rho}_0 > \hat{\rho}'_0 > \hat{\rho}_1 > \hat{\rho}'_1 > \cdots > \hat{\rho}_n > \hat{\rho}'_n > \hat{\rho}_{n+1} > \hat{\rho}'_{n+1} > 15\varepsilon_2$$

Now, choose $0 < \varepsilon_4$ sufficiently small so that the following $2n+2$ inequalities hold:

$$(90) \quad \hat{\rho}_k > \rho_n(\hat{\rho}_k, 10\varepsilon_4) > \hat{\rho}'_k + \varepsilon_2/100 \quad , \quad \hat{\rho}'_k > \rho_n(\hat{\rho}'_k, 10\varepsilon_4) > \hat{\rho}_{k+1} + \varepsilon_2/100 \quad ; \quad 1 \leq k \leq n+1$$

The full set of these inequalities are used in the proofs of Propositions 15.1 and 15.2, where they are multiplied by the scale $s = \lambda_{\mathcal{F}}^*/10$.

Finally, ε_0 is the “basic error” appearing in almost every transverse translation calculation and estimate, so is restricted by multiple conditions. The following constructions might be informally summarized by saying “it is intuitively clear that there exists ε_0 *sufficiently small* so that all of these conditions are satisfied”. The following specification of ε_0 makes this precise, as well as offering an argument that none of the details are particularly intuitively clear, except when the leaves are 1-dimensional. We list the requirements:

- (1) $\varepsilon_0 < 1/2000$ – used in equations (131) and (132)
- (2) $\varepsilon_0 \leq 50n(2/5)^n \varepsilon_1$ – used in equation (123)
- (3) $\varepsilon_0 < \varepsilon_2/2000$ – used in equations (128) and (129) and in proof of Proposition 15.1
- (4) $\varepsilon_0 < \varepsilon_3/4$ – used in equations (135), (156), (163) and in proof of Proposition 15.2
- (5) $\varepsilon_0 < \varepsilon_1/2 < \varepsilon_1 - 2\varepsilon_3$ – used in (173)
- (6) $\varepsilon_0 < \varepsilon_3/2 \{1 + 35n^{3/2} \cdot (4/15\varepsilon_2)^{n-1}\}$ – used in equation (153)
- (7) $\varepsilon_0 < \varepsilon_4/20$ – used in Proposition 13.4
- (8) $\varepsilon_0 < \delta_n(\varepsilon_4)/100$ for δ_n defined in Lemma 13.5

(91) Choose $\varepsilon_0 > 0$ to satisfy the 8 conditions above.

Recall that λ_{ε_0} was defined in Definition 12.4, so we introduce the fundamental “leafwise” constant:

$$(92) \quad \lambda_{\mathcal{F}}^* = \min\{\delta_{\mathcal{U}}^{\mathcal{F}}, \lambda_{\mathcal{F}}/5, \lambda_{\varepsilon_0}, 1\}$$

which is the basic distance scale for all of our subsequent constructions, chosen so that the leafwise balls $D_{\mathcal{F}}(\xi, \lambda_{\mathcal{F}}^*)$ are “ ε_0 -approximately Euclidean”. For example, if the leaves of \mathcal{F} are isometric to Euclidean space \mathbb{R}^n , then $\lambda_{\mathcal{F}}^* = \min\{\delta_{\mathcal{U}}^{\mathcal{F}}, \lambda_{\mathcal{F}}/5, 1\}$. Otherwise, if the leaves of \mathcal{F} have large sectional curvatures, then $\lambda_{\mathcal{F}}^*$ may be quite small.

Recall the definitions of the functions *var* in (70) and *div* in (72), and choose the “transverse” scale constant $r_* > 0$ so that $\text{div}(z, r_*) \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$ for all $z \in \mathfrak{M}$, and also $\text{var}(i, r_*) \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$ for all $1 \leq i \leq \nu$.

We next give several applications of these choices to the study of the “micro-local geometry” of a matchbox manifold. The following notion and estimates will be used frequently.

DEFINITION 13.1. *Let $\{X, d_X\}$ and $\{Y, d_Y\}$ be metric spaces, and $\epsilon > 0$. A homeomorphism into $\phi: X \rightarrow Y$ is said to be an ϵ -isometry if*

$$(93) \quad d_X(x, x') - \epsilon \leq d_Y(\phi(x), \phi(x')) \leq d_X(x, x') + \epsilon \quad \text{for all } x, x' \in X$$

LEMMA 13.2. *For $x \in \mathfrak{M}$ and orthonormal frame \hat{u} for $T_x \mathcal{F}$, the geodesic normal coordinate map $\varphi_{x, \hat{u}}^g: D_{\mathbb{R}^n}(\lambda_{\mathcal{F}}^*/2) \rightarrow D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)$ is an $\varepsilon_0 \lambda_{\mathcal{F}}^*$ -isometry from the metric $\|\cdot\|$ to the metric $d_{\mathcal{F}}$.*

Proof. The disk $D_{\mathbb{R}^n}(\lambda_{\mathcal{F}}^*/2)$ has diameter $\lambda_{\mathcal{F}}^*$, so it follows from the estimate $\|\widehat{g}^{\widehat{u}} - \delta\|_{\widehat{u}} \leq \varepsilon_0$ as in the proof of Lemma 12.2. \square

LEMMA 13.3. *Let $x \in U_i$ and $y \in \mathcal{P}_i(x) \cap U_j$ for some $1 \leq i, j \leq \nu$. Assume that $x' \in \mathfrak{S}(x, i, r_*)$ and $y' = \mathfrak{S}(y, j, r_*) \cap \mathcal{P}_i(x')$. Then*

$$(94) \quad d_{\mathcal{F}}(x, y) - 2\varepsilon_0 \lambda_{\mathcal{F}}^* \leq d_{\mathcal{F}}(x', y') \leq d_{\mathcal{F}}(x, y) + 2\varepsilon_0 \lambda_{\mathcal{F}}^*$$

If either $i = j$ or $x = y$, then a more strict estimate holds:

$$(95) \quad d_{\mathcal{F}}(x, y) - \varepsilon_0 \lambda_{\mathcal{F}}^* \leq d_{\mathcal{F}}(x', y') \leq d_{\mathcal{F}}(x, y) + \varepsilon_0 \lambda_{\mathcal{F}}^*$$

Proof. Let $y'' = \mathfrak{S}(y, i, r_*) \cap \mathcal{P}_i(x')$. Then $d_{\mathcal{F}}(y', y'') \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$ by the definition of the divergence (72) and of r_* . Thus, $|d_{\mathcal{F}}(x', y'') - d_{\mathcal{F}}(x', y')| \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$ by the triangle inequality.

Then by the definition of the variation (70) and r_* we also have $|d_{\mathcal{F}}(x', y'') - d_{\mathcal{F}}(x, y)| \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$. The estimates (94) and (95) then follow. \square

Finally, a more delicate estimate compares the local *affine* geometry of geodesic coordinates in nearby plaques. Let $x \in D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*) \subset U_i$ and $y \in D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)$ so that $D_{\mathcal{F}}(y, \lambda_{\mathcal{F}}^*/2) \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*)$.

Let $x' \in \mathfrak{S}(x, i, r_*)$ and set $y' = \phi_i(x, x')(y)$. Choose orthonormal frames \widehat{u} for $T_x \mathcal{F}$ and \widehat{v}' for $T_{y'} \mathcal{F}$, with corresponding geodesic coordinates $\varphi_{x, \widehat{u}}^g$ and $\varphi_{y', \widehat{v}'}^g$. Consider the composition

$$(96) \quad \Psi'_{x, y'} \equiv (\varphi_{y', \widehat{v}'}^g)^{-1} \circ \phi_i(x, x') \circ \varphi_{x, \widehat{u}}^g \circ T_{\xi}: D_{\mathbb{R}^n}(\lambda_{\mathcal{F}}^*) \rightarrow \mathbb{R}^n$$

where $\xi = (\varphi_{x, \widehat{u}}^g)^{-1}(y)$, and $T_{\xi}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the affine transformation defined by $T_{\xi}(\vec{a}) = \vec{a} + \xi$. Then

$$\Psi'_{x, y'}(\vec{0}) = (\varphi_{y', \widehat{v}'}^g)^{-1} \circ \phi_i(x, x') \circ \varphi_{x, \widehat{u}}^g(\xi) = (\varphi_{y', \widehat{v}'}^g)^{-1} \circ \phi_i(x, x')(y) = (\varphi_{y', \widehat{v}'}^g)^{-1}(y') = \vec{0}$$

Thus, the map $\Psi'_{x, y'}$ compares two coordinate systems about the point y' : one is the translate of the coordinates $\varphi_{x, \widehat{u}}^g$ centered at x but restricted to a neighborhood of y in its domain; and the other is centered at the translated point y' . The next result shows that $\Psi_{x, y'}$ can be made “almost the identity” by the proper choice of \widehat{v}' .

PROPOSITION 13.4. *There exists a choice of orthonormal frame \widehat{v} for $T_y \mathcal{F}$ so that*

$$(97) \quad \Psi_{x, y'} \equiv (\varphi_{y', \widehat{v}}^g)^{-1} \circ \phi_i(x, x') \circ \varphi_{x, \widehat{u}}^g \circ T_{\xi}: D(\lambda_{\mathcal{F}}^*/2) \rightarrow \mathbb{R}^n$$

is $\varepsilon_4 \lambda_{\mathcal{F}}^$ -close to the identity, for ε_4 defined by (90).*

Proof. The idea of the proof is simple, in that we express both maps $\varphi_{x, \widehat{u}}^g$ and $\varphi_{y', \widehat{v}'}^g$ in the local coordinates φ_i , as in the proof of Proposition 12.3. Then the transverse translation map $\phi_i(x, x')$ becomes the identity map, and the issue becomes simply how to choose the new framing \widehat{v} and obtaining an estimate for the error.

Let $\varphi_i(x) = (\tilde{x}, w_x)$ and $\varphi_i(y') = (\tilde{y}', w_{y'})$ for $w_x, w_{y'} \in \mathfrak{T}_i$. Then $y' = \phi_i(x, x')(y)$ implies $\tilde{y}' = \tilde{y}$.

Set $d_2 = \lambda_{\mathcal{F}}^*/5$. The restriction $\phi_i(x, x'): D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2) \rightarrow \mathcal{P}_i(w_{y'})$ is an ε_0 -isometry by Lemma 13.2, and thus $\phi_i(x, x')(D_{\mathcal{F}}(x, 2d_2)) \subset D_{\mathcal{F}}(y', \lambda_{\mathcal{F}}^*/2)$, so the composition (97) is well-defined. Let

$$\begin{aligned} \widetilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2) &= \varphi_i(D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)) \subset (-1, 1)^n \times \{w_x\} \\ \widetilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2) &= \varphi_i(D_{\mathcal{F}}(y', \lambda_{\mathcal{F}}^*/2)) \subset (-1, 1)^n \times \{w_{y'}\} \end{aligned}$$

Recall that \widetilde{d} denotes the distance function on $\widetilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2)$ defined by the leafwise metric $d_{\mathcal{F}}$ via the coordinates φ_i , and \widetilde{g} denotes the induced Riemannian metric on $\widetilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2)$. The geodesic coordinates associated to \widetilde{g} and \widehat{u} , centered at \tilde{x} , are denoted by $\widetilde{\exp}_{\tilde{x}, \widehat{u}}: D(\lambda_{\mathcal{F}}^*/2) \rightarrow \widetilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2)$. For $\xi = (\varphi_{x, \widehat{u}}^g)^{-1}(y)$ then $\tilde{y} = \widetilde{\exp}_{\tilde{x}, \widehat{u}}(\xi)$ and we have $\varphi_i(\tilde{y}) = y$.

Similarly, \tilde{d}' denotes the distance function induced on $\tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2)$, and \tilde{g}' denotes the induced metric tensor on $\tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2)$. The geodesic coordinates associated to \tilde{g}' and \tilde{v}' , centered at \tilde{y}' , are denoted by $\widetilde{\exp}_{\tilde{y}', \tilde{v}'} : D(\lambda_{\mathcal{F}}^*/2) \rightarrow \tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2)$. Then $\Psi_{x, y'} = \widetilde{\exp}_{\tilde{y}', \tilde{v}'}^{-1} \circ \widetilde{\exp}_{\tilde{x}, \tilde{u}} \circ T_{\xi}$ and we have:

$$(98) \quad \begin{array}{ccc} T_x \mathcal{F} \cong \mathbb{R}^n \supset D(\lambda_{\mathcal{F}}^*/2) & \xrightarrow{\widetilde{\exp}_{\tilde{x}, \tilde{u}} \circ T_{\xi}} & \tilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2) \subset (-1, 1)^n \times \{w_x\} \xrightarrow{\varphi_i^{-1}} \mathcal{P}_i(w_x) \\ \Psi_{x, y'} \downarrow & & = \downarrow \downarrow \phi_i(x, x') \\ T_{y'} \mathcal{F} \cong \mathbb{R}^n \supset D(\lambda_{\mathcal{F}}^*/2) & \xrightarrow{\widetilde{\exp}_{\tilde{y}', \tilde{v}'}} & \tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2) \subset (-1, 1)^n \times \{w_{y'}\} \xrightarrow{\varphi_i^{-1}} \mathcal{P}_i(w_{y'}) \end{array}$$

Set $\vec{\gamma} = \vec{x} + \hat{u} \cdot \xi$ so that $T_{\vec{\gamma}} \circ F_{\hat{u}}(\vec{a}) = T_{\vec{x}} \circ F_{\hat{u}}(\vec{a} + \xi)$.

By condition (78) of Definition 12.1, and using that $\lambda_{\mathcal{F}}^* \leq \lambda_{\epsilon_0}$, for all $\vec{a} \in D(\lambda_{\mathcal{F}}^*/2)$ we have

$$(99) \quad \tilde{d}(\widetilde{\exp}_{\tilde{x}, \tilde{u}}(\vec{a} + \xi), T_{\vec{\gamma}} \circ F_{\hat{u}}(\vec{a})) \leq \varepsilon_0 \lambda_{\mathcal{F}}^*, \quad \tilde{d}'(\widetilde{\exp}_{\tilde{y}', \tilde{v}'}(\vec{a}), T_{\vec{\gamma}} \circ F_{\hat{v}'}(\vec{a})) \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$$

Set $\vec{a} = \vec{0}$ in the first estimate of (99), then by Lemma 12.2 we obtain

$$(100) \quad \|\tilde{y} - \gamma\|_{\hat{u}} \leq \tilde{d}(\tilde{y}, \gamma) + \varepsilon_0 \lambda_{\mathcal{F}}^* = \tilde{d}(\widetilde{\exp}_{\tilde{x}, \tilde{u}}(\xi), T_{\vec{\gamma}}(\vec{0})) + \varepsilon_0 \lambda_{\mathcal{F}}^* \leq 2\varepsilon_0 \lambda_{\mathcal{F}}^*$$

The obvious next step is to replace the orthonormal framing \hat{v}' for \mathbb{R}^n for the norm $\|\cdot\|_{\hat{v}'}$ with the new framing $\hat{v} = \hat{u}$, as the claim of Proposition 13.4 would then follow easily. However, \hat{u} need not be an orthonormal framing the norm $\|\cdot\|_{\hat{v}'}$, so it is necessary to adjust the framing \hat{u} using the Gram-Schmidt orthogonalization process. This introduces additional errors, which depend on the “distance” from \hat{u} to \hat{v}' in the Lie group $GL(\mathbb{R}^n)$. We formulate this error as follows, using an elementary fact from linear algebra about continuity of the Gram-Schmidt orthogonalization process.

LEMMA 13.5. *Let \mathbb{R}^n have the standard Euclidean inner product with norm $\|\cdot\|$. There exists $\epsilon_n > 0$ and a monotone continuous function $\delta_n : [0, \epsilon_n] \rightarrow [0, \epsilon_n]$ with $\delta_n(0) = 0$, such that given $0 < \epsilon \leq \epsilon_n$ set $\delta = \delta_n(\epsilon) > 0$, then for any basis $\{\vec{f}'_1, \dots, \vec{f}'_n\} \subset \mathbb{R}^n$, whose vectors satisfy*

- (1) $1 - \delta < \|\vec{f}'_j\| < 1 + \delta$, for $1 \leq j \leq n$,
- (2) $|\vec{f}'_i \bullet \vec{f}'_j| < \delta$, for $1 \leq i \neq j \leq n$,

then there exists orthonormal vectors $\{\vec{f}_1, \dots, \vec{f}_n\}$ such that $\|\vec{f}_j - \vec{f}'_j\| \leq \epsilon$. □

Recall that $\hat{e} = \{\vec{e}_1, \dots, \vec{e}_n\}$ is the standard orthogonal basis for \mathbb{R}^n . Scale these unit vectors by a factor of d_2 so they lie in the domain of $\Psi_{x, y'}$, and set $\vec{z}_j = F_{\hat{u}}(d_2 \vec{e}_j)$. Note that $\|\vec{z}_j\|_{\hat{u}} = d_2$.

Then $d_2 \vec{e}_j + \xi \in \tilde{D}_i(\vec{x}, \lambda_{\mathcal{F}}^*/2)$, and set $\tilde{z}_j = \widetilde{\exp}_{\tilde{x}, \tilde{u}}(d_2 \vec{e}_j + \xi) \in \tilde{D}_i(\vec{x}, \lambda_{\mathcal{F}}^*/2)$. Then by Lemma 13.2,

$$(101) \quad |\tilde{d}(\tilde{z}_j, \tilde{y}) - d_2| = |\tilde{d}(\tilde{z}_j, \tilde{y}) - \|\tilde{z}_j\|_{\hat{u}}| = |\tilde{d}(\tilde{z}_j, \tilde{y}) - \|\hat{u} \cdot (d_2 \vec{e}_j + \vec{y}) - \hat{u} \cdot \vec{y}\|_{\hat{u}}| \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$$

$$(102) \quad |\tilde{d}(\tilde{z}_j, \tilde{z}_k) - \sqrt{2}d_2| = |\tilde{d}(\tilde{z}_j, \tilde{z}_k) - \|\hat{u} \cdot (d_2 \vec{e}_j + \vec{y}) - \hat{u} \cdot (d_2 \vec{e}_k + \vec{y})\|_{\hat{u}}| \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$$

The estimates (101) and (102) imply that the set $\{(\tilde{z}_1 - \tilde{y})/d_2, \dots, (\tilde{z}_n - \tilde{y})/d_2\}$ is an “almost orthonormal” collection for the metric \tilde{d} .

By Lemma 13.3, the map $\phi_i(x, x')$ is an $\varepsilon_0 \lambda_{\mathcal{F}}^*$ -isometry, and as $\phi_i(x, x')$ is the identity map in the coordinates φ_i , we have the corresponding estimates to (101) and (102) for the metric \tilde{d}' ,

$$(103) \quad |\tilde{d}'(\tilde{z}_j, \tilde{y}) - d_2| \leq 2\varepsilon_0 \lambda_{\mathcal{F}}^* \implies \|\tilde{z}_j - \tilde{y}\|_{\hat{v}'} - d_2 \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

$$(104) \quad |\tilde{d}'(\tilde{z}_j, \tilde{z}_k) - \sqrt{2}d_2| \leq 2\varepsilon_0 \lambda_{\mathcal{F}}^* \implies \|\tilde{z}_j - \tilde{z}_k\|_{\hat{v}'} - \sqrt{2}d_2 \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

where the implications follow from estimate (82) of Lemma 12.2.

Define $\vec{z}'_j = \widetilde{\exp}_{\tilde{y}', \tilde{v}'}^{-1}(\tilde{z}_j)$. Then by estimate (83) of Lemma 12.2, and noting that $\vec{z}'_j \in \tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2)$,

$$(105) \quad \|\tilde{z}_j - T_{\tilde{y}'} \circ F_{\tilde{v}'}(\vec{z}'_j)\|_{\hat{v}'} = \|\widetilde{\exp}_{\tilde{y}', \tilde{v}'}(\vec{z}'_j) - T_{\tilde{y}'} \circ F_{\tilde{v}'}(\vec{z}'_j)\|_{\hat{v}'} \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$$

Then by (103) and (104), and using that the map $T_{\tilde{y}'} \circ F_{\hat{v}'}$ is an isometry from the norm $\|\cdot\|$ to the norm $\|\cdot\|_{\hat{v}'}$, we obtain for the Euclidean norm on \mathbb{R}^n , for $1 \leq j \neq k \leq n$,

$$(106) \quad \|\tilde{z}'_j\| - d_2 \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

$$(107) \quad \|\tilde{z}'_j - \tilde{z}'_k\| - \sqrt{2}d_2 \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

Set $\vec{f}'_j = \tilde{z}'_j/d_2$, and observe that (106) implies the collection $\{\vec{f}'_1, \dots, \vec{f}'_n\}$ satisfies hypothesis (1) of Lemma 13.5 for $\delta = 15\varepsilon_0$.

It remains to estimate $|\vec{f}'_j \bullet \vec{f}'_k|$ for $1 \leq j \neq k \leq n$. Write $\vec{f}'_k = \vec{f}'_{j,k} + \vec{f}'_{k,k}$ where $\vec{f}'_{j,k}$ is collinear with \vec{f}'_j and $\vec{f}'_j \bullet \vec{f}'_{k,k} = 0$. Then $|\vec{f}'_j \bullet \vec{f}'_k| = |\vec{f}'_j \bullet \vec{f}'_{j,k}| = \|\vec{f}'_j\| \|\vec{f}'_{j,k}\|$.

Note also that $\|\vec{f}'_j\|^2 = \|\vec{f}'_{k,k}\|^2 + \|\vec{f}'_{j,k}\|^2$ hence $\|\vec{f}'_{j,k}\|^2 = (\|\vec{f}'_j\|^2 - \|\vec{f}'_{k,k}\|^2)$.

By (107) we have

$$(108) \quad \sqrt{2} - 15\varepsilon_0 \leq \|\vec{f}'_j - \vec{f}'_k\| = \|(\vec{f}'_j - \vec{f}'_{j,k}) - \vec{f}'_{k,k}\| \leq \sqrt{2} + 15\varepsilon_0$$

After squaring and using the orthogonality of the vectors, we obtain

$$2 - 30\sqrt{2}\varepsilon_0 + 225\varepsilon_0^2 \leq \|(\vec{f}'_j - \vec{f}'_{j,k})\|^2 + \|\vec{f}'_{k,k}\|^2 \leq 2 + 30\sqrt{2}\varepsilon_0 + 225\varepsilon_0^2$$

Then note $\varepsilon_0 < 1/10,000$ implies $2 - 100\varepsilon_0 < 2 - 30\sqrt{2}\varepsilon_0 + 225\varepsilon_0^2$. Also, \vec{f}'_j and $\vec{f}'_{j,k}$ are collinear, hence $\|\vec{f}'_j - \vec{f}'_{j,k}\|^2 = \|\vec{f}'_j\|^2 - 2\|\vec{f}'_j\| \cdot \|\vec{f}'_{j,k}\| + \|\vec{f}'_{j,k}\|^2$. Thus, we have

$$2 - 100\varepsilon_0 < \|\vec{f}'_j\|^2 - 2\|\vec{f}'_j\| \cdot \|\vec{f}'_{j,k}\| + \|\vec{f}'_{j,k}\|^2 + \|\vec{f}'_{k,k}\|^2 < 2 + 100\varepsilon_0$$

From the identity $\|\vec{f}'_{k,k}\|^2 = (\|\vec{f}'_j\|^2 - \|\vec{f}'_{j,k}\|^2)$ one derives $\|\vec{f}'_j \bullet \vec{f}'_{j,k}\| = \|\vec{f}'_{j,k}\| < 100\varepsilon_0$.

Thus, the collection $\{\vec{f}'_1, \dots, \vec{f}'_n\}$ satisfies both hypotheses of Lemma 13.5 for $\delta = 100\varepsilon_0$. We assume that $\varepsilon_0 < \varepsilon_4/20$ in (91) so that $\varepsilon_4 < \varepsilon_5 = 2\varepsilon_4 - 20\varepsilon_0$. By choice of ε_0 in (91), we have $100\varepsilon_0 < \delta_n(\varepsilon_5)$ so we obtain the orthonormal framing $\hat{f} = \{\vec{f}_1, \dots, \vec{f}_n\}$ of \mathbb{R}^n satisfying $\|\vec{f}_k - \vec{f}'_k\| \leq \varepsilon_5$.

Define $\vec{v}_j = F_{\hat{v}'}(\vec{f}_j)$, then $\hat{v} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal frame for the norm $\|\cdot\|_{\hat{v}'}$ so defines an orthonormal framing of $T_{\tilde{y}'}\mathcal{F}$. Note that $F_{\hat{v}} = F_{\hat{v}'} \circ F_{\hat{f}}$ and calculate:

$$\begin{aligned} & \|T_{\tilde{\gamma}} \circ F_{\hat{u}}(d_2 \vec{e}_j) - T_{\tilde{y}'} \circ F_{\hat{v}}(d_2 \vec{e}_j)\|_{\hat{v}'} \\ & \leq \| (T_{\tilde{\gamma}} \circ F_{\hat{u}}(d_2 \vec{e}_j) - \tilde{z}_j) \|_{\hat{v}'} + \|\tilde{z}_j - T_{\tilde{y}'} \circ F_{\hat{v}}(d_2 \vec{e}_j)\|_{\hat{v}'} \\ & = \|T_{\tilde{\gamma}} \circ F_{\hat{u}}(d_2 \vec{e}_j) - \tilde{z}_j\|_{\hat{v}'} + \|\tilde{z}_j - T_{\tilde{y}'} \circ F_{\hat{v}'}(d_2 \vec{f}'_j)\|_{\hat{v}'} \\ & \leq \|T_{\tilde{\gamma}} \circ F_{\hat{u}}(d_2 \vec{e}_j) - \tilde{z}_j\|_{\hat{v}'} + \|\tilde{z}_j - T_{\tilde{y}'} \circ F_{\hat{v}'}(d_2 \vec{f}'_j)\|_{\hat{v}'} + \|T_{\tilde{y}'} \circ F_{\hat{v}'}(d_2 \vec{f}'_j) - T_{\tilde{y}'} \circ F_{\hat{v}}(d_2 \vec{f}'_j)\|_{\hat{v}'} \\ & \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_0 \lambda_{\mathcal{F}}^* + 2d_2 \varepsilon_5 = 4\varepsilon_0 \lambda_{\mathcal{F}}^* + 2\varepsilon_5 \lambda_{\mathcal{F}}^*/5 \end{aligned}$$

where we use successively the definitions of the quantities involved, Lemmas 12.2 and 13.2, the estimate (99), the estimate (105), and Lemma 13.5.

Then by the approximations (99) and Lemmas 12.2 and 13.2, we have for all $\vec{a} \in D(2\lambda_{\mathcal{F}}^*/5)$ that

$$(109) \quad \tilde{d}(\widetilde{\text{exp}}_{\vec{x}, \hat{u}}(\vec{a} + \xi), \widetilde{\text{exp}}_{\tilde{y}', \hat{v}'}(\vec{a})) \leq 7\varepsilon_0 \lambda_{\mathcal{F}}^* + 2\varepsilon_5 \lambda_{\mathcal{F}}^*/5$$

Hence by Lemma 13.2 and our choice $\varepsilon_5 = 2\varepsilon_4 - 20\varepsilon_0$, we obtain

$$(110) \quad \|\Psi_{x, y'}(\vec{a}) - \vec{a}\| = \|\widetilde{\text{exp}}_{\tilde{y}', \hat{v}'}^{-1} \circ \widetilde{\text{exp}}_{\vec{x}, \hat{u}} \circ T_{\xi}(\vec{a})\| \leq 8\varepsilon_0 \lambda_{\mathcal{F}}^* + 2\varepsilon_5 \lambda_{\mathcal{F}}^*/5 < \varepsilon_4 \lambda_{\mathcal{F}}^*$$

completing the proof of Proposition 13.4. \square

The fine control of the affine structure of geodesic coordinates provided by Proposition 13.4 is used in establishing robustness criteria for leafwise Delaunay triangulations in the next section. We define a “non-linear” form of the robustness criteria in Definition 10.5 and Proposition 11.3.

Recall that $\text{Span}(\vec{v}_0, \dots, \vec{v}_k) \subset \mathbb{R}^n$ is the *affine* span of the vectors $\{\vec{v}_0, \dots, \vec{v}_k\}$.

DEFINITION 13.6. Let $\rho > 0$ and $x \in U_i$ such that $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*) \subset \mathcal{P}_i(x)$. Let $1 \leq m \leq n$. A set $\{y_0, \dots, y_m\} \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)$ is ρ -robust if the following leafwise metric conditions hold, for each $1 \leq k < m$:

- (1) Fix an orthonormal frame $\hat{u} = \{\vec{u}_1, \dots, \vec{u}_n\} \subset T_{y_k}\mathcal{F}$ with geodesic coordinates $\varphi_{y_k, \hat{u}}^g$;
- (2) for each $0 \leq j \leq k$, set $\vec{v}_j = (\varphi_{y_k, \hat{u}}^g)^{-1}(y_j)$;
- (3) Set $H(y_0, \dots, y_k; y_k) = \varphi_{y_k, \hat{u}}^g \{\text{Span}(\vec{v}_0, \dots, \vec{v}_k) \cap D_{\mathbb{R}^n}(\lambda_{\mathcal{F}}^*)\}$.

Then the point y_{k+1} lies at distance at least ρ from the submanifold $H(y_0, \dots, y_k; y_k)$

We now show that the robustness condition for points in Definition 13.6 implies the robustness condition Definition 10.5 holds for their vector coordinates in geodesic normal coordinates.

PROPOSITION 13.7. *Given constants*

$$(111) \quad \lambda_{\mathcal{F}}^*/10 = d_1 < d_1 + 2\varepsilon_0\lambda_{\mathcal{F}}^* < e_1 < e_2 < d_2 - 2\varepsilon_0\lambda_{\mathcal{F}}^* < d_2 = 2\lambda_{\mathcal{F}}^*/10$$

and $0 < \rho_0 < d_1$, let $x \in \mathfrak{M}$ and suppose $\{y_0, \dots, y_m\} \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)$ satisfy

- (1) $e_1 \leq d_{\mathcal{F}}(y_j, y_k) \leq 2e_2$ for $0 \leq j \neq k \leq m$
- (2) $\{y_0, \dots, y_m\}$ is ρ_0 -robust.

Given an orthonormal frame \hat{u} of $T_x\mathcal{F}$, set $\vec{w}_j = (\varphi_{x, \hat{u}}^g)^{-1}(y_j) \in D(\lambda_{\mathcal{F}}^*/2)$ for $0 \leq j \leq m$. Then $\{\vec{w}_0, \dots, \vec{w}_m\} \subset \mathbb{R}^n$ is ρ_m -robust, where $\rho_\ell = \rho_\ell(\rho_0, \varepsilon_4\lambda_{\mathcal{F}}^*, d_1, d_2)$ is defined by (88) for $1 \leq \ell \leq m$.

Proof. We proceed by induction. Fix $x \in \mathfrak{M}$. By assumption, $d_{\mathcal{F}}(y_0, y_1) \geq e_1 > d_1 + \varepsilon_0\lambda_{\mathcal{F}}^*$, so by Lemma 13.2 we have $\|\vec{w}_1 - \vec{w}_0\| \geq d_1 \geq \rho_0 - 2\varepsilon_4\lambda_{\mathcal{F}}^* = \rho_1$.

Now assume that the collection $\{\vec{w}_0, \dots, \vec{w}_\ell\}$ is ρ_ℓ -robust, for each $1 \leq \ell < m$. We show that $\{\vec{w}_0, \dots, \vec{w}_{\ell+1}\}$ is $\rho_{\ell+1}$ -robust.

Let U_i be a coordinate chart such that $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*) \subset U_i$.

Let $T_{\vec{w}_\ell}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T_{\vec{w}_\ell}(\vec{x}) = \vec{x} + \vec{w}_\ell$ be translation by \vec{w}_ℓ . Define the composition

$$\Psi_\ell \equiv (\varphi_{y_\ell, \hat{v}}^g)^{-1} \circ \varphi_{x, \hat{u}}^g \circ T_{\vec{w}_\ell}: D(2d_2) \rightarrow D(\lambda_{\mathcal{F}}^*/2)$$

for an orthonormal frame $\hat{v} = \hat{v}_\ell$ of $T_{y_\ell}\mathcal{F}$ as provided by Proposition 13.4 so that $\|\Psi_\ell(\vec{x}) - \vec{x}\| \leq \varepsilon_4\lambda_{\mathcal{F}}^*$, where ε_4 is defined by (90). This is possible by our choice of ε_0 in (91). (In this application of Proposition 13.4, we take $y' = y_\ell \in \mathcal{P}_i(x)$ which is on the same plaque as x .)

For $0 \leq j \leq m$, define $\vec{z}_j = (\varphi_{y_\ell, \hat{v}}^g)^{-1}(y_j)$, and also set $\vec{w}'_j = \vec{w}_j - \vec{w}_\ell$. Then $\Psi_\ell(\vec{w}'_j) = \vec{z}_j$. Using that Ψ_ℓ is $\varepsilon_4\lambda_{\mathcal{F}}^*$ close to the identity, we have that each $\|\vec{z}_j - \vec{w}'_j\| \leq \varepsilon_4\lambda_{\mathcal{F}}^*$.

Note that $\{\vec{w}_0, \dots, \vec{w}_\ell\}$ is ρ_ℓ -robust if and only if the collection $\{\vec{w}'_0, \dots, \vec{w}'_\ell\}$ is ρ_ℓ -robust.

By the definition that $\{y_0, \dots, y_m\}$ is ρ_0 -robust, the point $y_{\ell+1}$ lies at distance at least ρ_0 from the submanifold $\varphi_{y_\ell, \hat{v}}^g(\text{Span}(\vec{z}_0, \dots, \vec{z}_\ell))$. So by Lemma 13.2, the vector $\vec{z}_{\ell+1}$ lies at distance at least $\rho_0 - \varepsilon_0\lambda_{\mathcal{F}}^* \geq \rho_\ell$ from the linear span $\text{Span}(\vec{z}_0, \dots, \vec{z}_\ell)$. Thus, we also have that the collection $\{\vec{z}_0, \dots, \vec{z}_{\ell+1}\}$ is ρ_ℓ -robust.

It is given that $e_1 \leq d_{\mathcal{F}}(y_j, y_k) \leq e_2$ for $0 \leq j \neq k \leq m$, so by Lemma 13.2 we have

$$d_1 < e_1 - 2\varepsilon_0\lambda_{\mathcal{F}}^* \leq \|\vec{z}_j - \vec{z}_k\| \leq 2e_2 + 2\varepsilon_0\lambda_{\mathcal{F}}^* < 2d_2$$

for all $0 \leq j \neq k \leq m$. We can thus apply Proposition 11.3 for $\varepsilon = \varepsilon_4\lambda_{\mathcal{F}}^*$ to the collection $\{\vec{z}_0, \dots, \vec{z}_{\ell+1}\}$ to conclude that $\text{Span}(\vec{w}_0, \dots, \vec{w}_{\ell+1})$ is $\rho_{\ell+1}$ -robust. \square

PART V - CONSTRUCTIONS OF TRANSVERSE FOLIATIONS

In this last Part V, we give the construction of a nice stable transversal \mathcal{X} , for all of \mathfrak{M} in the equicontinuous case, or on \mathfrak{N} in the “Big Box” case. The main theorems then follow by Proposition 9.2. We first consider the equicontinuous case, where the transversal is inductively defined in Section 14, and it is proven to be stable in Section 15. The modifications of these arguments in Sections 14 and 15 required to show Theorem 1.2 are considered in Section 16.

14. A NICE STABLE TRANSVERSAL

Recall that the constant $\epsilon_{\mathcal{U}}^{\mathcal{F}}$ is the “leafwise Lebesgue number” defined in equation (5), so that for all $y \in \mathfrak{M}$, $D_{\mathcal{F}}(y, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset D_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/2)$. Fix a basepoint $w_0 \in \mathfrak{T}_{i_1}$, set $x_0 = \tau_1(w_0) = \varphi_1^{-1}(0, w_0) \in U_1$ and let L_0 be the leaf through x_0 with holonomy covering \tilde{L}_0 .

Recall from section 7 that $\mathcal{N}_0 \subset L_0$ is a net for L_0 which is $\epsilon_{\mathcal{U}}^{\mathcal{F}}/4$ -spanning, and that $x_0 \in \mathcal{N}_0$. The lift of \mathcal{N}_0 to \tilde{L}_0 defines a net $\tilde{\mathcal{N}}_0$ containing the lift \tilde{x}_0 of x_0 to the constant path.

Recall that the constant $\lambda_{\mathcal{F}}^*$ was defined by (92), and r_* determined by this choice. Define a sequence of constants based on the scale $\lambda_{\mathcal{F}}^*$.

$$(112) \quad \begin{aligned} d_1 &= .10 \cdot \lambda_{\mathcal{F}}^*, & d'_1 &= .11 \cdot \lambda_{\mathcal{F}}^*, & d''_1 &= .12 \cdot \lambda_{\mathcal{F}}^* \\ d_2 &= .20 \cdot \lambda_{\mathcal{F}}^*, & d'_2 &= .19 \cdot \lambda_{\mathcal{F}}^*, & d''_2 &= .18 \cdot \lambda_{\mathcal{F}}^* \end{aligned}$$

Note that

$$\lambda_{\mathcal{F}}^*/10 = d_1 < d'_1 < d''_1 < d''_2 < d'_2 < d_2 = \lambda_{\mathcal{F}}^*/5$$

For each $z \in \mathcal{N}_0$, the index $1 \leq i_z \leq \nu$ is chosen so that $B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$. For $z = x_0$ we take $i_{x_0} = 1$, so $B_{\mathfrak{M}}(x_0, \epsilon_{\mathcal{U}}) \subset U_1$. For $\tilde{z} \in \tilde{\mathcal{N}}_0$ with projection $z = \Pi(\tilde{z})$, we set $i_{\tilde{z}} = i_z$.

Recall that for $\tilde{z} \in \tilde{\mathcal{N}}_0$ then $h_{\tilde{z}}$ denotes the holonomy along some nice path $\tilde{\gamma}$ from \tilde{x}_0 to \tilde{z} , considered as a transformation of the transverse space $\mathfrak{T}_* \subset \mathfrak{X}$.

Now assume that the pseudogroup $\mathcal{G}_{\mathcal{F}}^*$ acting on \mathfrak{T}_* is equicontinuous, and let $\delta_0^{\mathcal{T}}$ denote the constant of equicontinuity for $\mathcal{G}_{\mathcal{F}}^*$ for $\epsilon = r_*/2$. Then by Theorem 4.8, there exists a $\mathcal{G}_{\mathcal{F}}$ -invariant clopen subset $w_0 \in V \subset \mathfrak{T}_*$ satisfying $\text{diam}_{\mathfrak{X}}(V_{\tilde{z}}) < \delta_0^{\mathcal{T}}$ for all $\tilde{z} \in \tilde{\mathcal{N}}_0$. For all $\tilde{z} \in \tilde{\mathcal{N}}_0$, we then have

$$V_{\tilde{z}} \subset B_{\mathfrak{X}}(w_{\tilde{z}}, r_*/2) \subset \mathfrak{T}_{i_{\tilde{z}}}$$

Hence, for any $w \in V_{\tilde{z}}$ we have $V_{\tilde{z}} \subset B_{\mathfrak{X}}(w, r_*)$.

For each $\tilde{z} \in \tilde{\mathcal{N}}_0$, define the compact sets

$$(113) \quad \mathfrak{U}_{\tilde{z}}^V = \pi_z^{-1}(V_{\tilde{z}}) \subset U_{i_{\tilde{z}}} \quad , \quad \tilde{\mathfrak{U}}_{\tilde{z}}^V = \mathfrak{U}_{\tilde{z}}^V \times \{\tilde{z}\} \subset \tilde{U}_{i_{\tilde{z}}}$$

For $x \in \mathfrak{U}_{\tilde{z}}^V$, define a *standard section* by

$$(114) \quad \mathfrak{S}(x, i_{\tilde{z}}, V) = \varphi_{i_{\tilde{z}}}^{-1}(\lambda_{i_{\tilde{z}}}(x), V_{\tilde{z}}) \subset \mathfrak{U}_{\tilde{z}}^V \subset \tilde{\mathfrak{U}}_{\tilde{z}}^V \quad , \quad \tilde{\mathfrak{S}}(x, i_{\tilde{z}}, V) \equiv \mathfrak{S}(x, i_{\tilde{z}}, V) \times \{\tilde{z}\}$$

The following concept will be used in the inductive construction of \mathcal{X} .

DEFINITION 14.1. *Given a leaf $L \subset \mathfrak{M}$, a subset $Y \subset L$, and $R > 0$, define the penumbra of Y of thickness R by*

$$(115) \quad \text{Pen}_{\mathcal{F}}(Y, R) = \{y \in L \mid d_{\mathcal{F}}(y, Y) \leq R\}$$

That is, $\text{Pen}_{\mathcal{F}}(Y, R)$ is the closed subset of L consisting of all points within distance R of Y .

The translates of the set V are indexed by the points $\tilde{z} \in \tilde{\mathcal{N}}_0$. We introduce a trick to simplify the presentation below somewhat. By Theorem 4.10, the leaves without holonomy form a dense subset of \mathfrak{M} . Thus, we can assume the basepoint $w_0 \in V$ corresponds to a leaf L_0 without holonomy. That is, the covering map $\tilde{L}_0 \rightarrow L_0$ is a diffeomorphism. Then each point $\tilde{z} \in \tilde{\mathcal{N}}_0$ corresponds to a

unique point $z \in \mathcal{N}_0$, so by an abuse of notation, we can index the holonomy maps $h_z = h_{\bar{z}}$ and the translates $V_z = V_{\bar{z}}$ by the points $z \in \mathcal{N}_0$.

The construction of a nice stable transversal \mathcal{X} proceeds by inductively choosing a net in the leaf L_0 . The net points $\xi_k \in L_0$ must satisfy appropriate net and general position conditions, and then we let \mathcal{X} be the union of the standard transversals $\mathcal{X}_k = \mathfrak{S}(\xi_k, i_{\xi_k}, V)$ through these points. The procedure for making these choices is rather straightforward, though we utilize the constants and estimates from previous sections. These choices ensure that our inductive procedure does not halt prematurely, and when complete, satisfies the stability condition.

We will construct a net in the unbounded complete manifold L_0 , so it is necessary to have a “stopping criterion”. This is provided by the following remarks. The leaf L_0 is dense in \mathfrak{M} , so there exists $R_1 > 0$ sufficiently large so that $D_{\mathcal{F}}(x_0, R_1) = \text{Pen}_{\mathcal{F}}(\{x_0\}, R_1)$ contains a plaque from every chart in the *finite* collection $\{\mathfrak{U}_z^V \mid z \in \mathcal{N}_0\}$.

We now begin the inductive construction of $\mathcal{X} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_{p_*} \subset \mathfrak{M}$. Set

$$(116) \quad \xi_1 = x_0, \quad \Xi_1 = \{\xi_1\}, \quad z_1 = x_0, \quad \Lambda_1 = \{z_1\}, \quad \theta_1 = i_{z_1} = 1$$

Define $\mathcal{X}_1 = \mathfrak{S}(\xi_1, \theta_1, V) \subset \mathfrak{M}$. Then $x_0 \in \mathcal{X}_1 \subset \mathfrak{U}_{\theta_1}^V$.

Set $\mathcal{M}_1(x_0) = \mathcal{X}_1 \cap L_0$ and $\widetilde{\mathcal{M}}_1(x_0) = \Pi^{-1}(\mathcal{M}_1(x_0))$.

Note that for $\xi \neq \xi' \in \mathcal{M}_1(x_0)$, ξ, ξ' lie in disjoint plaques of L_0 , hence $d_{\mathcal{F}}(\xi, \xi') \geq 2\delta_{\mathcal{U}}^{\mathcal{F}}$. It follows that $\mathcal{M}_1(x_0)$ is a net for L_0 which is R_1 -dense, and whose points are $2\delta_{\mathcal{U}}^{\mathcal{F}}$ -separated.

Now let $p > 1$, and assume we are given:

- (i) $\Xi_p = \{\xi_1, \dots, \xi_p\} \subset L_0$
- (ii) $\Lambda_p = \{z_1, \dots, z_p\} \subset \mathcal{N}_0$ such that $\xi_j \in B_{\mathcal{F}}(z_j, \epsilon_{\mathcal{U}}^{\mathcal{F}}/2)$.
- (iii) $\theta_j = i_{z_j}$ for $1 \leq j \leq p$
- (iv) $\mathcal{X}_j = \mathfrak{S}(\xi_j, \theta_j, \ell_0)$ for $1 \leq j \leq p$.

Then set $\widehat{\mathcal{X}}_p = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_p$ and $\mathcal{M}_p = \widehat{\mathcal{X}}_p \cap L_0$. Assume that

- (v) For all $y \neq z \in \mathcal{M}_p$, then $d_{\mathcal{F}}(y, z) \geq d'_1$.

For $n \geq 2$, to ensure that we obtain a transversal \mathcal{X} which is regular and stable, we require two additional requirements for the inductive choice of ξ_{p+1} , and thus of $\widehat{\mathcal{X}}_{p+1}$, which will ensure that the n -simplices of the triangulation associated to the nets $\mathcal{M}(x)$ satisfy uniform distribution and stability conditions.

Let $\Delta_{\mathcal{F}}(\mathcal{M}_p)$ denote the *partial leafwise simplicial complex* of \mathcal{M}_p . This follows the definition of the Delaunay triangulation given previously, but with a key restriction. A $(k+1)$ -tuple $\{y_0, \dots, y_k\} \subset \mathcal{M}_p$ defines a k -simplex $\Delta(y_0, \dots, y_k) \in \Delta_{\mathcal{F}}(\mathcal{M}_p)$ if there exists $\omega \in L_0$ and $0 < r \leq d'_2$ such that $B_{\mathcal{F}}(\omega, r) \cap \widehat{\mathcal{X}}_p = \emptyset$ and $\{y_0, \dots, y_k\} \subset S_{\mathcal{F}}(\omega, r) \cap \widehat{\mathcal{X}}_p$. Even though the net \mathcal{M}_p need not be d'_2 -dense, we still restrict the inscribed spheres to diameter at most d'_2 .

Let $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}(\mathcal{M}_p)$ be an n -simplex, then by Proposition 8.3, the collection of hyperplanes $\{L(y_0, y_i) \mid 1 \leq i \leq n\}$ in $D_{\mathcal{F}}(y_0, \lambda_{\mathcal{F}})$ have non-trivial intersection:

$$(117) \quad \omega(y_0, \dots, y_n) = L(y_0, y_1) \cap \dots \cap L(y_0, y_n)$$

The point $\omega(y_0, \dots, y_n) \in L_0$ is the center of the associated inscribed closed disk with radius $r(y_0, \dots, y_n) = d_{\mathcal{F}}(y_\ell, \omega(y_0, \dots, y_n))$ for all $0 \leq \ell \leq n$.

The transversal $\widehat{\mathcal{X}}_p$ is then assumed to satisfy:

- (vi) For all $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{M}_p)$ and $\xi \in \mathcal{M}_p - \{y_0, \dots, y_n\}$,
- $$(118) \quad d_{\mathcal{F}}(\xi, \omega(y_0, \dots, y_n)) \geq r(y_0, \dots, y_n) + \varepsilon_1 \lambda_{\mathcal{F}}^*$$

In particular, this implies that $\Delta_{\mathcal{F}}^{(n+1)}(\mathcal{M}_p)$ is empty.

Introduce the constant $R_p > 0$ which is the largest radius R such that

$$(119) \quad B_{\mathcal{F}}(x_0, R) \subset \text{Pen}_{\mathcal{F}}(\mathcal{M}_p, d_2'') = \bigcup_{z \in \mathcal{M}_p} D_{\mathcal{F}}(z, d_2'')$$

In particular, $B_{\mathcal{F}}(x_0, d_2'') = \text{Pen}_{\mathcal{F}}(z_1, d_2'') \subset \text{Pen}_{\mathcal{F}}(\mathcal{M}_p, d_2'')$, so $R_p \geq d_2''$. If no maximum exists, that is $\text{Pen}_{\mathcal{F}}(\mathcal{M}_p, d_2'') = L_0$, then the inductive process terminates.

Assume that $R_p < \infty$, then by the definition of R_p we can choose

$$(120) \quad \xi'_{p+1} \in D_{\mathcal{F}}(x_0, R_p + d_1/2) \cap \{\text{Pen}_{\mathcal{F}}(\mathcal{M}_p, d_2'') - \text{Pen}_{\mathcal{F}}(\mathcal{M}_p, d_1'')\}$$

This choice guarantees that $d_{\mathcal{F}}(\xi'_{p+1}, z) > d_1''$ for all $z \in \mathcal{M}_p$.

The requirement that $\xi'_{p+1} \in D_{\mathcal{F}}(x_0, R_p + d_1/2)$ is made so that the next point added to \mathcal{M}_p is “almost as close as possible” to $x_0 \in L_0$. This procedure also makes it obvious that the induction terminates in a finite number of steps.

Next, choose $z_{p+1} \in \mathcal{N}_0 \cap B_{\mathcal{F}}(\xi'_{p+1}, \epsilon_{\mathcal{U}}^{\mathcal{F}}/2)$, which is possible by the assumption that \mathcal{N}_0 is a net which is $\epsilon_{\mathcal{U}}^{\mathcal{F}}/2$ -dense. Set $\theta_{p+1} = i_{z_{p+1}}$.

It remains to modify the choice of ξ'_{p+1} to a point $\xi_{p+1} \in B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ satisfying the inductive conditions defined below.

Consider the disk $D_{\mathcal{F}}(\xi'_{p+1}, 4d_2) \subset B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*) \subset B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}})$. Introduce the set

$$(121) \quad \Omega(\xi'_{p+1}) = D_{\mathcal{F}}(\xi'_{p+1}, 4d_2) \cap \mathcal{M}_p$$

Since the points of \mathcal{M}_p are d_1' -separated, and $d_2 = 2d_1$, the metric conditions (78–80) and a standard volume estimate yields that the cardinality of $\Omega(\xi'_{p+1})$ is at most 10^n .

Let $\Omega^{(n)}(\xi'_{p+1}) \subset \Delta_{\mathcal{F}}^{(n)}(\mathcal{M}_p)$ be the subset of all n -simplices whose vertices are contained in $\Omega(\xi'_{p+1})$. The cardinality of the set $\Omega^{(n)}(\xi'_{p+1})$ is thus bounded above by the constant C_n defined in (84).

For each n -simplex $\Delta(y_0, \dots, y_n) \in \Omega^{(n)}(\xi'_{p+1})$ recall that $\omega(y_0, \dots, y_n) \in D_{\mathcal{F}}(\xi'_{p+1}, 4d_2)$ denotes the center of the inscribed sphere for its vertices, so $\{y_0, \dots, y_n\} \subset S_{\mathcal{F}}(\omega(y_0, \dots, y_n), r(y_0, \dots, y_n))$.

For each n -simplex $\Delta(y_0, \dots, y_n) \in \Omega^{(n)}(\xi'_{p+1})$ and constant $\kappa > 0$, form the annular region

$$A_{\mathcal{F}}(y_0, \dots, y_n; \kappa) = \text{Pen}_{\mathcal{F}}(S_{\mathcal{F}}(\omega(y_0, \dots, y_n), r(y_0, \dots, y_n)), \kappa)$$

LEMMA 14.2. *Let $\kappa = 2\varepsilon_1 \lambda_{\mathcal{F}}^*$. Then for an n -simplex $\Delta(y_0, \dots, y_n) \in \Omega^{(n)}(\xi'_{p+1})$,*

$$(122) \quad \text{Vol}_{\mathcal{F}} A_{\mathcal{F}}(y_0, \dots, y_n; \kappa) \leq 200 \cdot 2^n \varepsilon_1 (\lambda_{\mathcal{F}}^*/5)^n$$

Proof. Let Φ_n be the constant such that $\text{Vol}_{\hat{u}} D_{\mathbb{R}^n}(s) = \Phi_n s^n$. Note that $(\sqrt{2})^n \leq \Phi_n \leq 2^n$.

By the condition (80) for $0 < s \leq \lambda_{\mathcal{F}}^* \leq \lambda_{\varepsilon_0}$ we have

$$|\Phi_n s^n - \text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\omega(y_0, \dots, y_n), s)| \leq \varepsilon_0 \cdot s^n \leq \varepsilon_0 \cdot (\lambda_{\mathcal{F}}^*)^n$$

Hence, we have

$$\begin{aligned} & \text{Vol}_{\mathcal{F}} A_{\mathcal{F}}(y_0, \dots, y_n; \kappa) \\ &= |\text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\omega(y_0, \dots, y_n), r(y_0, \dots, y_n) + \kappa) - \text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\omega(y_0, \dots, y_n), r(y_0, \dots, y_n) - \kappa)| \\ &\leq \Phi_n \cdot \{(r(y_0, \dots, y_n) + \kappa)^n - (r(y_0, \dots, y_n) - \kappa)^n\} + 2\varepsilon_0 (\lambda_{\mathcal{F}}^*)^n \end{aligned}$$

Given $\kappa = 2\varepsilon_1 \lambda_{\mathcal{F}}^*$ with $20n\varepsilon_1 < 1$, and $\lambda_{\mathcal{F}}^*/10 \leq r \leq \lambda_{\mathcal{F}}^*/5$, elementary estimates yield

$$\begin{aligned} \{(r + \kappa)^n - (r - \kappa)^n\} &= r^n \cdot \{(1 + \kappa/r)^n - (1 - \kappa/r)^n\} \\ &\leq r^n \cdot \{(\exp(n\kappa/r) - \exp(-n\kappa/r))\} \\ &\leq (\lambda_{\mathcal{F}}^*/5)^n \cdot \{(\exp(20n \varepsilon_1) - \exp(-20n \varepsilon_1))\} \\ &\leq 100n \varepsilon_1 (\lambda_{\mathcal{F}}^*/5)^n \end{aligned}$$

Combining these estimates and (85), we obtain

$$\begin{aligned}
 \text{Vol}_{\mathcal{F}} A_{\mathcal{F}}(y_0, \dots, y_n, \kappa) &\leq \Phi_n \cdot \{(r(y_0, \dots, y_n) + \kappa)^n - (r(y_0, \dots, y_n) - \kappa)^n\} + 2\varepsilon_0(\lambda_{\mathcal{F}}^*)^n \\
 &\leq \{\Phi_n \cdot 100n \varepsilon_1 + 2 \cdot 5^n \varepsilon_0\}(\lambda_{\mathcal{F}}^*/5)^n \\
 &\leq \{2^n \cdot 100n \varepsilon_1 + 2 \cdot 5^n \varepsilon_0\}(\lambda_{\mathcal{F}}^*/5)^n \\
 (123) \quad &\leq 200n \cdot 2^n \varepsilon_1(\lambda_{\mathcal{F}}^*/5)^n \quad \square
 \end{aligned}$$

The total volume of all such annular regions intersecting $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/100)$ is bounded above by

$$C_n \cdot 200n \cdot 2^n \varepsilon_1(\lambda_{\mathcal{F}}^*/5)^n$$

We also have the estimate of the leafwise volume of the disk

$$|\Phi_n \cdot (\lambda_{\mathcal{F}}^*/200)^n - \text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)| \leq \varepsilon_0 \cdot (\lambda_{\mathcal{F}}^*/200)^n = (\varepsilon_0/2^n) \cdot (\lambda_{\mathcal{F}}^*/100)^n$$

so that

$$(124) \quad \text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200) \geq \Phi_n \cdot (\lambda_{\mathcal{F}}^*/200)^n - \varepsilon_0 \cdot (\lambda_{\mathcal{F}}^*/200)^n \geq (1/40)^n \cdot (\lambda_{\mathcal{F}}^*/5)^n$$

Now, given $\varepsilon_1 = 1/(C_n \cdot 1000n \cdot 100^n)$ by (85), it follows that

$$(125) \quad C_n \cdot 200n \cdot 2^n \varepsilon_1 \cdot (\lambda_{\mathcal{F}}^*/5)^n \leq \frac{1}{4} \cdot 1/40^n \cdot (\lambda_{\mathcal{F}}^*/5)^n$$

Thus, the total volume of all annular regions intersecting $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ is less than $1/4$ of its volume. Therefore, if we choose $\xi_{p+1} \in B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ which lies outside of this union of annular regions, then it will satisfy the restricted condition:

(vi') For all $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{M}_p)$

$$(126) \quad d_{\mathcal{F}}(\xi_{p+1}, \omega(y_0, \dots, y_n)) \geq r(y_0, \dots, y_n) + 2\varepsilon_1 \lambda_{\mathcal{F}}^*$$

Next, it is necessary to show condition (126) holds for all nets $\mathcal{M}_p(y)$, and not just for \mathcal{M}_p . To establish that this general condition holds, we must choose ξ_{p+1} to also satisfy a robustness condition, which is the point of the following additional restriction.

For $1 \leq k < n$, let $\{y_0, \dots, y_k\} \subset \Omega(\xi'_{p+1})$ be a collection of distinct points with $y_k \in \mathcal{X}_{i_k}$ where $i_0 < \dots < i_k \leq p$.

Let $\hat{u} = \{\vec{u}_1, \dots, \vec{u}_n\} \subset T_{y_k} \mathcal{F}$ be an orthonormal frame, and $\varphi_{y_k, \hat{u}}^g: D(\lambda_{\mathcal{F}}^*) \rightarrow D_{\mathcal{F}}(y_k, \lambda_{\mathcal{F}}^*) \subset L_0$ the corresponding geodesic coordinates. Define $\vec{y}_j = (\varphi_{y_k, \hat{u}}^g)^{-1}(y_j)$ for $0 \leq j \leq k+1$. Note that $\vec{y}_k = \vec{0}$.

Let $\text{Span}(\vec{y}_0, \dots, \vec{y}_k) \subset \mathbb{R}^n$ be the linear submanifold through the origin of dimension k which they span. Then define a submanifold of $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$,

$$(127) \quad H(y_0, \dots, y_k; \xi'_{p+1}) = \varphi_{y_k, \hat{u}}^g \{ \text{Span}(\vec{y}_0, \dots, \vec{y}_k) \cap D(2d_2) \} \cap D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$$

which has diameter at most $\lambda_{\mathcal{F}}^*/100$, and thus has $(n-1)$ -volume bounded above by $(\lambda_{\mathcal{F}}^*/100)^{n-1}$.

Form the $2\varepsilon_2 \lambda_{\mathcal{F}}^*$ -thickening of $H(y_0, \dots, y_k; \xi'_{p+1})$,

$$(128) \quad \mathcal{S}(y_0, \dots, y_k; \xi'_{p+1}, 2\varepsilon_2 \lambda_{\mathcal{F}}^*) = \text{Pen}_{\mathcal{F}}(H(y_0, \dots, y_k; \xi'_{p+1}), 2\varepsilon_2 \lambda_{\mathcal{F}}^*) \cap D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$$

Then by the estimate (81) and $\varepsilon_0 \leq \varepsilon_2$, its volume is bounded above by

$$(129) \quad 4(\varepsilon_2 \lambda_{\mathcal{F}}^*) \cdot (\lambda_{\mathcal{F}}^*/100)^{n-1} + \varepsilon_0(\lambda_{\mathcal{F}}^*/100)^n \leq 5(\varepsilon_2 \lambda_{\mathcal{F}}^*) \cdot (\lambda_{\mathcal{F}}^*/100)^{n-1}$$

The total number of such submanifolds $H(y_0, \dots, y_k; \xi'_{p+1})$ in $D_{\mathcal{F}}(\xi'_{p+1}, 4d_2)$ is bounded above by the constant C_n from (84), hence the total volume of all such sets which intersect $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ is thus bounded above by

$$(130) \quad C_n \cdot 5(\varepsilon_2 \lambda_{\mathcal{F}}^*) \cdot (\lambda_{\mathcal{F}}^*/100)^{n-1} = \frac{C_n \cdot (5\lambda_{\mathcal{F}}^*) \cdot (\lambda_{\mathcal{F}}^*/100)^{n-1}}{C_n \cdot 2000 \cdot 2^n} = \frac{1}{4} \cdot 1/40^n \cdot (\lambda_{\mathcal{F}}^*/5)^n$$

where we use the definition of ε_2 in (86).

Thus, the total volume of all slabs intersecting $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ is less than $1/4$ of its volume, so we may choose $\xi_{p+1} \in B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ which is disjoint from the union of all annular and slab regions introduced above. Then set:

- $\Xi_{p+1} = \{\xi_1, \dots, \xi_{p+1}\} \subset L_0$, $\Lambda_{p+1} = \{z_1, \dots, z_{p+1}\} \subset \mathcal{N}_0$
- $\mathcal{X}_{p+1} = \mathfrak{S}(\xi_{p+1}, \theta_{p+1}, \ell_0)$, $\hat{\mathcal{X}}_{p+1} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_p \cup \mathcal{X}_{p+1}$
- $\mathcal{M}_{p+1} = \hat{\mathcal{X}}_{p+1} \cap L_0$

Now proceed in this manner until for $p = p_*$ we have $\text{Pen}_{\mathcal{F}}(\mathcal{M}_{p_*}, d_2'') = L_0$. Then set

$$\mathcal{X} \equiv \hat{\mathcal{X}}_{p_*} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_{p_*}, \quad \Xi \equiv \Xi_{p_*}, \quad \mathcal{M} \equiv \mathcal{M}_{p_*} = \mathcal{X} \cap L_0$$

We also set $\mathcal{M}(x) = L_x \cap \mathcal{X}$ and $\mathcal{M}_p(x) = L_x \cap \hat{\mathcal{X}}_p$.

By previous remarks, the transversal \mathcal{X} is invariant, as it is the union of holonomy invariant sections of the form $\mathfrak{S}(\xi_p, \theta_p, \ell_0)$. The proof that \mathcal{X} is uniform follows from Lemmas 14.3 and 14.4 below.

LEMMA 14.3. *For all $x \in \mathfrak{M}$ and $y \neq z \in \mathcal{M}(x)$ we have $d_{\mathcal{F}}(y, z) \geq .114 \cdot \lambda_{\mathcal{F}}^* > d_1'$.*

Proof. Let $y \neq z \in \mathcal{M}(x)$. Then by definition, there exists $\xi_i, \xi_j \in \Xi_{p_*}$ for $1 \leq i, j \leq p_*$ such that $y \in \mathfrak{S}(\xi_i, \theta_i, \ell_0) \cap L_x$ and $z \in \mathfrak{S}(\xi_j, \theta_j, \ell_0) \cap L_x$. Without loss of generality we can assume that $i \geq j$.

If $z \notin \mathcal{P}_{\theta_i}(y)$ then $d_{\mathcal{F}}(y, z) \geq \delta_{\mathcal{U}}^{\mathcal{F}} > \lambda_{\mathcal{F}}^* > d_1'$. Thus, we may assume that $z \in \mathcal{P}_{\theta_i}(y)$, so $i > j$.

Set $y' = \mathfrak{S}(\xi_i, \theta_i, \ell_0) \cap \mathcal{P}_{\theta_i}(\xi_i) = \xi_i$ and set $z' = \mathfrak{S}(\xi_j, \theta_j, \ell_0) \cap \mathcal{P}_{\theta_i}(\xi_i)$.

Then $d_{\mathcal{F}}(y', z') \geq d_1'' - \lambda_{\mathcal{F}}^*/200 = .115 \cdot \lambda_{\mathcal{F}}^*$ by the choice of ξ_i' satisfying (120) and the choice of ξ_i .

Apply Lemma 13.3 for the pairs $\{y, z\}$ and $\{y', z'\}$ and note that $2\varepsilon_0 < 1/1000$ to obtain

$$(131) \quad d_{\mathcal{F}}(y, z) \geq d_{\mathcal{F}}(y', z') - 2\varepsilon_0 \cdot \lambda_{\mathcal{F}}^* > .115 \cdot \lambda_{\mathcal{F}}^* - .001 \cdot \lambda_{\mathcal{F}}^* = .114 \cdot \lambda_{\mathcal{F}}^*$$

Thus, for all $x \in \mathfrak{M}$ the net $\mathcal{M}(x)$ is $(.114 \cdot \lambda_{\mathcal{F}}^*)$ -separated. \square

LEMMA 14.4. *For $x \in \mathfrak{M}$, and $y \in L_x$ there exists $z \in \mathcal{M}(x)$ such that $d_{\mathcal{F}}(y, z) \leq .181 \cdot \lambda_{\mathcal{F}}^* < d_1'$.*

Proof. For $x \in \mathfrak{M}$, let $y \in L_x$. Let $1 \leq i \leq p_*$ be such that $y \in \mathfrak{U}_{\theta_i}^{\ell_0}$ and thus $y = \mathcal{P}_{\theta_i}(y) \cap \mathfrak{S}(y, \theta_i, \ell_0)$.

Choose $\zeta \in \mathcal{X}_i \cap L_0$ and set $y' = \mathcal{P}_{\theta_i}(\zeta) \cap \mathfrak{S}(y, \theta_i, \ell_0) \in L_0$.

We are given that $\text{Pen}_{\mathcal{F}}(\mathcal{M}, d_2'') = L_0$, so there exists $\xi \in \mathcal{M}$ such that $d_{\mathcal{F}}(y', \xi) \leq d_2'' = .118 \cdot \lambda_{\mathcal{F}}^*$.

By definition of \mathcal{M} , there exists $\xi_j \in \Xi$ for some $1 \leq j \leq p_*$ such that $\xi = \mathcal{P}_{\theta_i}(\xi) \cap \mathfrak{S}(\xi_j, \theta_i, \ell_0)$.

Then let $z = \mathcal{P}_{\theta_i}(y) \cap \mathfrak{S}(\xi_j, \theta_i, \ell_0) \in \mathcal{M}(x)$, and apply Lemma 13.3 for the pairs $\{y, z\}$ and $\{y', \xi\}$ to obtain

$$(132) \quad d_{\mathcal{F}}(y, z) \leq d_{\mathcal{F}}(y', \xi) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* \leq d_2'' + .001 \cdot \lambda_{\mathcal{F}}^* = .181 \cdot \lambda_{\mathcal{F}}^*$$

Thus, for all $x \in \mathfrak{M}$ the net $\mathcal{M}(x)$ is $.181 \cdot \lambda_{\mathcal{F}}^*$ -dense. \square

15. PROOF OF STABILITY

It remains to show that the transversal \mathcal{X} is regular and stable. These properties are at the heart of the construction of the transverse foliation \mathcal{H} from a Delaunay triangulation in the proof of Proposition 9.2. At first inspection, stability of simplices for a Delaunay triangulation associated to a net $\mathcal{M}(x)$ seems to be intuitively clear, and in fact this is basically correct for dimension $n \leq 2$. The difficulty is that for $n > 2$, as x varies, the “small variations” of the points of $\mathcal{M}(x)$ may result in an abrupt change in the Delaunay simplicial structure, if any face of a Voronoi cell has too small of a diameter relative to the size of the variation. Propositions 15.1, 15.2 and 15.5 which follow, show this does not happen for the nets defined by the section \mathcal{X} .

We first establish some notation used in the demonstrations. Let $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$. By permuting the order of the vertices, we can assume that there exists $1 \leq i_0 < i_1 < \dots < i_n \leq p_*$ and points $\xi_{i_k} \in \Xi_{p_*} \subset L_0$ such that $y_k = \mathcal{X}_{i_k} \cap \mathcal{P}_{\theta_{i_n}}(y_n)$.

For $x_n \in \mathcal{X}_{i_n} \subset \mathcal{U}_{\theta_{i_n}}^{\ell_0}$ let $\mathcal{P}_n(x_n) = \mathcal{P}_{\theta_{i_n}}(x_n)$ denote the plaque containing x_n in the chart φ_{i_n} .

Set $x_k = \mathcal{X}_{i_k} \cap \mathcal{P}_n(x_n)$ for $0 \leq k \leq n$. We must show that $\Delta(x_0, \dots, x_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$. The proof of this proceeds in three steps.

The first step is to show that the robustness condition is stable. Recall that robustness for a set of vectors $\{\vec{v}_0, \dots, \vec{v}_m\}$ was defined in Definition 10.5, and in Proposition 11.3 we showed that robustness is stable under perturbations of sets of vectors. Definition 13.6 defined robustness for a set of points $\{y_0, \dots, y_m\}$ in a leaf, and in Proposition 13.7 this condition was shown to imply robustness for the corresponding set of vectors defined by a geodesic coordinates. The following result shows that robustness for a set of points in a leaf is stable under transverse translation of the set to nearby leaves. The proof of this uses all of the previous methods as cited, and is probably the most subtle technical result of this Appendix. The conclusion is fundamental for the proof of the subsequent two results which establish that \mathcal{X} is stable, Propositions 15.2 and 15.5.

Note that the proof below uses an induction procedure which invokes Proposition 13.7 repeatedly, and the constants defined by (89) and the inequalities (90) defining ε_4 . As promised earlier, for $1 \leq \ell \leq n$, introduce the scaled constants $\tilde{\rho}_\ell = \hat{\rho}_\ell \lambda_{\mathcal{F}}^*/10$ and $\tilde{\rho}'_\ell = \hat{\rho}'_\ell \lambda_{\mathcal{F}}^*/10$.

PROPOSITION 15.1. *For all $x_n \in \mathcal{X}_{i_n}$, the collection $\{x_0, \dots, x_n\}$ is $3\varepsilon_2 \lambda_{\mathcal{F}}^*/2$ -robust.*

Proof. We proceed via induction on $1 \leq m < n$. Note that $\tilde{\rho}_0 = 18\varepsilon_2 \lambda_{\mathcal{F}}^*/10$.

The first step of the induction, $m = 1$, is trivial. Given $\{x_0, x_1\} \subset \mathcal{P}_n(\xi)$ as above, with $x_\ell \in \mathcal{X}_{i_\ell}$ and $i_0 \neq i_1$ then $d_{\mathcal{F}}(x_1, x_0) \geq d'_1$ by Lemma 14.3, and $d'_1 \geq 2\varepsilon_2 \lambda_{\mathcal{F}}^* > \tilde{\rho}_0 > \tilde{\rho}_1$ hence $\{x_0, x_1\}$ is $\tilde{\rho}_1$ -robust.

We make an inductive hypothesis which is uniform for all simplices. That is, for fixed $m < n$, assume that for all $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$, then for all subsets of points $\{x_0, \dots, x_m\}$ defined for $x_n \in \mathcal{X}_{i_m}$ as above, the set $\{x_0, \dots, x_m\}$ is $\tilde{\rho}_m$ -robust. We then show that each transverse translate of $\{y_0, \dots, y_{m+1}\}$ is $\tilde{\rho}_{m+1}$ -robust.

Consider first the case $z_{m+1} = \xi_{i_{m+1}} \in \mathcal{X}_{i_{m+1}}$, and set $z_k = \mathcal{X}_{i_k} \cap \mathcal{P}_n(\xi_{i_{m+1}})$ for $0 \leq j \leq n$. By the inductive hypothesis, the set $\{z_0, \dots, z_m\}$ is $\tilde{\rho}_m$ -robust. We verify the conditions of Definition 13.6 for the vertex, z_{m+1} . The point $\xi_{i_{m+1}}$ was chosen so that it lies outside of all $2\varepsilon_2 \lambda_{\mathcal{F}}^*$ -neighborhoods as defined in (128) of the images under the exponential map of affine subspaces spanned by local collections of at most $n + 1$ points. It follows, in particular, that the distance from $\xi_{i_{m+1}}$ to the submanifold $H(z_0, \dots, z_m; z_m)$ in Definition 13.6.3 is at least $2\varepsilon_2 \lambda_{\mathcal{F}}^* > \tilde{\rho}_m$. Thus, $\{z_0, \dots, z_{m+1}\}$ is also $\tilde{\rho}_m$ -robust.

Note that $d'_1 \leq d_{\mathcal{F}}(z_j, z_k)$ for $0 \leq j \neq k \leq n$ by Lemma 14.3. We are given $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$ which implies that the vertices $\{y_0, \dots, y_n\}$ admit an inscribed sphere, which must have radius at most $.181 \cdot \lambda_{\mathcal{F}}^*$ by Lemma 14.4. Thus, $d_{\mathcal{F}}(y_j, y_k) \leq .362 \cdot \lambda_{\mathcal{F}}^*$. The map ϕ_{y_n, z_n} is an $\varepsilon_0 \lambda_{\mathcal{F}}^*$ -isometry by Lemma 13.3, so $d_{\mathcal{F}}(z_j, z_k) \leq .362 \cdot \lambda_{\mathcal{F}}^* + \varepsilon_0 \cdot \lambda_{\mathcal{F}}^* < .380 \cdot \lambda_{\mathcal{F}}^* = 2d'_2$. It follows that the set of points $\{z_0, \dots, z_n\}$ satisfy the hypotheses of Proposition 13.7 for $e_1 = d'_1$, $e_2 = d'_2$, and $\rho = \tilde{\rho}_m$.

For simplicity, set $\zeta = z_{m+1}$, and choose an orthonormal frame \hat{u} of $T_{\zeta}\mathcal{F}$. Then for $0 \leq j \leq m$, let $\vec{z}_j = (\varphi_{\zeta, \hat{u}}^g)^{-1}(z_j)$ for the geodesic coordinates $\varphi_{\zeta, \hat{u}}^g: D(\lambda_{\mathcal{F}}^*) \rightarrow D_{\mathcal{F}}(\zeta, \lambda_{\mathcal{F}}^*)$. Then the collection $\{\vec{z}_0, \dots, \vec{z}_{m+1}\} \subset \mathbb{R}^n$ is $\tilde{\rho}'_m$ -robust by Proposition 13.7 and the choice of ε_4 in (90).

The robustness for the set $\{\vec{z}_0, \dots, \vec{z}_{m+1}\}$ is used to show it for $\{x_0, \dots, x_{m+1}\}$. Set $\zeta' = x_{m+1}$, then by Proposition 13.4, there exists an orthonormal framing \hat{v} of $T_{\zeta'}\mathcal{F}$ so that the composition

$$\Psi_{\zeta, \zeta'} \equiv (\varphi_{\zeta', \hat{v}}^g)^{-1} \circ \phi_i(\zeta, \zeta') \circ \varphi_{\zeta, \hat{u}}^g \circ T_{\zeta}: D(\lambda_{\mathcal{F}}^*/2) \rightarrow \mathbb{R}^n \cong T_{\zeta'}\mathcal{F}$$

is $\varepsilon_4 \lambda_{\mathcal{F}}^*$ -close to the identity. Set $\vec{w}_j = (\varphi_{\zeta', \hat{v}}^g)^{-1}(x_k) = \Psi_{\zeta, \zeta'}(\vec{z}_j)$, then $\|\vec{w}_j - \vec{z}_j\| \leq \varepsilon_4 \lambda_{\mathcal{F}}^*$.

The set $\{\vec{w}_1, \dots, \vec{w}_{m+1}\}$ satisfies the hypotheses of Proposition 11.3 for $e_1 = d_1$, $e_2 = d_2$, $\varepsilon = \varepsilon_4 \lambda_{\mathcal{F}}^*$ and $\rho = \tilde{\rho}'_m$. Therefore, $\{\vec{w}_1, \dots, \vec{w}_{m+1}\}$ is $(\tilde{\rho}_{m+1} + \varepsilon_2 \lambda_{\mathcal{F}}^*/1000)$ -robust.

Finally, by Lemma 13.2 the geodesic map $\varphi_{\zeta', \hat{v}}^g$ is an $\varepsilon_0 \lambda_{\mathcal{F}}^*$ -isometry, hence the distance from x_{m+1} to the submanifold $H(x_0, \dots, x_m; x_m)$ in Definition 13.6.3 is at least $\tilde{\rho}_{m+1} + \varepsilon_2 \lambda_{\mathcal{F}}^*/1000 - \varepsilon_0 \lambda_{\mathcal{F}}^* > \tilde{\rho}_{m+1}$.

This completes the inductive step. It remains to note that $\tilde{\rho}_n > 3\varepsilon_2 \lambda_{\mathcal{F}}^*/2$ by definition (89). \square

The next step towards showing that \mathcal{X} is regular and stable is to show:

PROPOSITION 15.2. *For all $x_n \in \mathcal{X}_{i_n}$, there exists $\omega(x_0, \dots, x_n) \in \mathcal{P}_n(x_n)$ and $r(x_0, \dots, x_n)$ such that*

$$(133) \quad \{x_0, \dots, x_n\} \subset S_{\mathcal{F}}(\omega(x_0, \dots, x_n), r(x_0, \dots, x_n)) \cap \mathcal{M}(x_n)$$

Moreover, the center satisfies, for ε_3 defined by (87),

$$(134) \quad d_{\mathcal{F}}(\omega(x_0, \dots, x_n), \omega'(y_0, \dots, y_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

where $\omega'(y_0, \dots, y_n) = \phi_{i_n}(y_n, x_n)(\omega(y_0, \dots, y_n))$ is the translate for the center of the inscribed sphere for the n -simplex $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$. In particular, this implies

$$(135) \quad |r(x_0, \dots, x_n) - r(y_0, \dots, y_n)| < \varepsilon_3 \lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* < \varepsilon_3 \lambda_{\mathcal{F}}^*$$

Proof. By rearranging the order of the vertices if necessary, we may assume that there are indices $i_0 < i_1 < \dots < i_n \leq p_*$ and points $\xi_{i_k} \in \Xi_{p_*} \subset L_0$ such that $y_k = \mathcal{X}_{i_k} \cap \mathcal{P}_n(y_n)$ for $0 \leq k \leq n$.

Let $\omega = \omega(y_0, \dots, y_n) \in \mathcal{P}_n(y_n)$ denote the center of the inscribed sphere for $\{y_0, \dots, y_n\}$, and let $r(y_0, \dots, y_n)$ denote its radius. Then $d'_1/2 \leq r(y_0, \dots, y_n) \leq d'_2$ as $\mathcal{M}(y_n)$ is d'_2 -dense and d'_1 -separated. Note that this implies $\{y_0, \dots, y_n, \omega\} \subset D(y_n, \lambda_{\mathcal{F}}^*/2)$.

By Proposition 15.1, the set $\{x_0, \dots, x_n\} \subset \mathcal{P}_n(x_n)$ is $\tilde{\rho}_n$ -robust.

Let $\phi_{i_n}(y_n, x_n): \mathcal{P}_n(y_n) \rightarrow \mathcal{P}_n(x_n)$ be the transverse transport map for the chart φ_{i_n} .

Let $\omega'(y_0, \dots, y_n) = \phi_{i_n}(y_n, x_n)(\omega)$ denote the translation of $\omega(y_0, \dots, y_n)$ to $\mathcal{P}_n(x_n)$.

For $0 \leq j \leq n$, we have the radius equalities $d_{\mathcal{F}}(y_j, \omega) = r(y_0, \dots, y_n)$, hence by Lemma 13.3,

$$(136) \quad r(y_0, \dots, y_n) - 2\varepsilon_0 \lambda_{\mathcal{F}}^* \leq d_{\mathcal{F}}(x_j, \omega'(y_0, \dots, y_n)) \leq r(y_0, \dots, y_n) + 2\varepsilon_0 \lambda_{\mathcal{F}}^*$$

Note that x_ℓ is the transverse transport of y_ℓ for the coordinate system φ_{i_ℓ} , while $\omega'(y_0, \dots, y_n)$ is the transport of $\omega(y_0, \dots, y_n)$ for the coordinate system φ_{i_n} and $i_\ell \neq i_n$. Thus, we must use (94) in place of the sharper estimate (95). Similarly, for $0 \leq j \neq k \leq n$, we have

$$(137) \quad d'_1 \leq d_{\mathcal{F}}(x_j, x_k) \leq 2d'_2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* < d_2$$

It follows that we also have $\{x_0, \dots, x_n, \omega'(y_0, \dots, y_n)\} \subset D(x_n, \lambda_{\mathcal{F}}^*/5)$.

The first step is to construct an inscribed sphere with center $\vec{\omega}(\vec{v}_0, \dots, \vec{v}_n)$ for the linearized problem in the tangent space $T_{x_n} \mathcal{F}$, and then modify the construction to obtain an inscribed sphere with center $\omega(x_0, \dots, x_n) \in \mathcal{P}_n(x_n)$ for the leafwise metric.

Choose $\xi \in \mathcal{P}_n(x_n)$ so that $\{x_0, \dots, x_n, \omega'(y_0, \dots, y_n)\} \subset B_{\mathcal{F}}(\xi, 2d_2)$. Let $\hat{u} = \{\vec{u}_1, \dots, \vec{u}_n\} \subset T_{\xi} \mathcal{F}$ be an orthonormal frame, with corresponding geodesic coordinates $\varphi_{\xi, \hat{u}}^g$ about ξ .

Set $\vec{v}_k = (\varphi_{\xi, \hat{u}}^g)^{-1}(x_k)$ for $0 \leq k \leq n$, then $\{\vec{v}_0, \dots, \vec{v}_n\} \subset \mathbb{R}^n$ is $\tilde{\rho}'_{n+1}$ -robust by Proposition 13.7.

Now set $\vec{\omega}'(y_0, \dots, y_n) = (\varphi_{\xi, \hat{u}}^g)^{-1}(\omega'(y_0, \dots, y_n))$. Then by Lemma 13.2 and (136) we have

$$(138) \quad r(y_0, \dots, y_n) - 3\varepsilon_0 \lambda_{\mathcal{F}}^* \leq \|\vec{v}_j - \vec{\omega}'(y_0, \dots, y_n)\| \leq r(y_0, \dots, y_n) + 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

while (137) implies, for $0 \leq j \neq k \leq n$,

$$(139) \quad d_1 < d'_1 - \varepsilon_0 \lambda_{\mathcal{F}}^* \leq \|\vec{v}_j - \vec{v}_k\| \leq 2d'_2 + 3\varepsilon_0 \lambda_{\mathcal{F}}^* < 2d_2$$

We can thus apply Proposition 11.2 for $e_1 = d_1$, $e_2 = d_2 = 2d_1$, $\rho = \tilde{\rho}'_{n+1} > 3\varepsilon_2\lambda_{\mathcal{F}}^*/2$ and $C_1 = 3\varepsilon_0\lambda_{\mathcal{F}}^*$ to conclude that there exists an inscribed sphere $S(\omega(\vec{v}_0, \dots, \vec{v}_n), r(\vec{v}_0, \dots, \vec{v}_n)) \subset D(\lambda_{\mathcal{F}}^*)$ such that

$$\begin{aligned} \|\omega(\vec{v}_0, \dots, \vec{v}_n) - \vec{\omega}'(y_0, \dots, y_n)\| &\leq 3\varepsilon_0 \cdot \left\{ n^{3/2} (2d_2)^{n-1} / (3\varepsilon_2\lambda_{\mathcal{F}}^*/2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^* \\ (140) \qquad \qquad \qquad &\leq 3\varepsilon_0 \cdot \left\{ n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^* \end{aligned}$$

That is, the vector $\omega(\vec{v}_0, \dots, \vec{v}_n)$ is a solution of the linearized problem of finding the center of an inscribed sphere, and (140) estimates the Euclidean distance to the translated center.

The task now is to convert this approximate answer to a solution for the leafwise metric. Let \tilde{d} denote the distance function on $D(\lambda_{\mathcal{F}}^*)$ induced from $d_{\mathcal{F}}$ by $\varphi_{\xi, \hat{u}}^g: D(\lambda_{\mathcal{F}}^*/2) \rightarrow L_{\xi}$. Then by Lemma 13.2,

$$(141) \qquad \qquad \qquad |\tilde{d}(\vec{a}, \vec{b}) - \|\vec{a} - \vec{b}\|| \leq \varepsilon_0 \lambda_{\mathcal{F}}^* \quad , \quad \text{for all } \vec{a}, \vec{b} \in D(\lambda_{\mathcal{F}}^*/2)$$

Introduce the equidistant submanifolds for the metric \tilde{d} ,

$$(142) \qquad \qquad \qquad \mathcal{H}(\vec{v}_j, \vec{v}_k) = \{\vec{z} \in D(\lambda_{\mathcal{F}}^*/2) \mid \tilde{d}(\vec{z}, \vec{v}_j) = \tilde{d}(\vec{z}, \vec{v}_k)\}$$

and the “thickened” equidistant sets for the leafwise metric, for $\epsilon > 0$,

$$(143) \qquad \qquad \qquad \mathcal{H}(\vec{v}_j, \vec{v}_k; \epsilon) = \{\vec{z} \in D(\lambda_{\mathcal{F}}^*/2) \mid -\epsilon \leq \tilde{d}(\vec{z}, \vec{v}_j) - \tilde{d}(\vec{z}, \vec{v}_k) \leq \epsilon\}$$

$$(144) \qquad \qquad \qquad \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; \epsilon) = \mathcal{H}(\vec{v}_0, \vec{v}_n; \epsilon) \cap \dots \cap \mathcal{H}(\vec{v}_{n-1}, \vec{v}_n; \epsilon)$$

Then (136) implies the translation $\vec{\omega}'(y_0, \dots, y_n) \in \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*)$, so this set is not empty. The key idea is to obtain a bound for its diameter, from which the proof of Proposition 15.2 follows. To this end, define the set of approximate solutions of the linearized problem by

$$(145) \qquad \qquad \qquad B(\vec{v}_0, \dots, \vec{v}_n; \epsilon) = \{\vec{z} \in D(\lambda_{\mathcal{F}}^*/2) \mid -\epsilon \leq \|\vec{z} - \vec{v}_j\| - \|\vec{z} - \vec{v}_n\| \leq \epsilon, \quad 0 \leq j < n\}$$

Note that the actual solution satisfies $\omega(\vec{v}_0, \dots, \vec{v}_n) \in B(\vec{v}_0, \dots, \vec{v}_n; \epsilon)$ for all $\epsilon > 0$.

LEMMA 15.3. $\mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; \epsilon) \subset B(\vec{v}_0, \dots, \vec{v}_n; \epsilon + 4\varepsilon_0\lambda_{\mathcal{F}}^*)$

Proof. Using (141) for $\vec{z} \in D(\lambda_{\mathcal{F}}^*/2)$, we have that

$$(146) \quad |(\|\vec{z} - \vec{v}_j\| - \|\vec{z} - \vec{v}_k\|) - 2\varepsilon_0\lambda_{\mathcal{F}}^*| \leq |\tilde{d}(\vec{z}, \vec{v}_j) - \tilde{d}(\vec{z}, \vec{v}_k)| \leq |(\|\vec{z} - \vec{v}_j\| - \|\vec{z} - \vec{v}_k\|)| + 2\varepsilon_0\lambda_{\mathcal{F}}^*$$

and the claim follows. \square

Thus, we now have $\vec{\omega}'(y_0, \dots, y_n) \in \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*) \subset B(\vec{v}_0, \dots, \vec{v}_n; 8\varepsilon_0\lambda_{\mathcal{F}}^*)$.

LEMMA 15.4. *Let $\vec{z} \in B(\vec{v}_0, \dots, \vec{v}_n; 8\varepsilon_0\lambda_{\mathcal{F}}^*)$, then*

$$(147) \qquad \qquad \qquad \|\vec{z} - \omega(\vec{v}_0, \dots, \vec{v}_n)\| \leq 32\varepsilon_0 \cdot \left\{ n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^*$$

Proof. Using the notation of Propositions 11.1 and 11.2, with \vec{v}_j in place of \vec{z}_j and \vec{z} in place of ω , and $e_1 = d_1$, $e_2 = d_2 = 2d_1$, $\rho = \tilde{\rho}'_{n+1} > 3\varepsilon_2\lambda_{\mathcal{F}}^*/2$ and $C_1 = 8\varepsilon_0\lambda_{\mathcal{F}}^*$, then $\vec{\zeta} = \vec{z} - \omega(\vec{v}_0, \dots, \vec{v}_n)$ is a solution of the matrix inequality

$$(148) \qquad \qquad \qquad \mathbf{V} \cdot \vec{\zeta} \in B(0, 2\sqrt{n} \cdot d_2 \cdot 8\varepsilon_0\lambda_{\mathcal{F}}^*)$$

Then by (56), we have the estimate $\|\mathbf{V}^{-1}\| \leq n \cdot (2d_2)^{n-1} / d_1 (3\varepsilon_2\lambda_{\mathcal{F}}^*/2)^{n-1}$ which yields

$$\begin{aligned} \|\vec{z} - \omega(\vec{v}_0, \dots, \vec{v}_n)\| &\leq \left\{ n \cdot (2d_2)^{n-1} / d_1 (3\varepsilon_2\lambda_{\mathcal{F}}^*/2)^{n-1} \right\} \cdot \{2\sqrt{n} \cdot d_2 \cdot 8\varepsilon_0\lambda_{\mathcal{F}}^*\} \\ &\leq \varepsilon_0 \cdot \left\{ 32n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^* \end{aligned}$$

where we use that $d_1 = \lambda_{\mathcal{F}}^*/10$ and $d_2 = 2\lambda_{\mathcal{F}}^*/10$ to simplify, yielding (147). \square

It follows from Lemmas 15.3 and 15.4 that the closed set $\mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*)$ is bounded, and is non-empty as $\vec{\omega}'(y_0, \dots, y_n) \in \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*)$. Thus, the intersection

$$(149) \quad \tilde{\omega}(\vec{v}_0, \dots, \vec{v}_n) = \mathcal{H}(\vec{v}_0, \vec{v}_n) \cap \dots \cap \mathcal{H}(\vec{v}_{n-1}, \vec{v}_n) \subset \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*)$$

is non-empty by transversality of the submanifolds $\mathcal{H}(\vec{v}_j, \vec{v}_n)$. Moreover, (147) implies that

$$(150) \quad \|\tilde{\omega}(\vec{v}_0, \dots, \vec{v}_n) - \omega(\vec{v}_0, \dots, \vec{v}_n)\| \leq 32\varepsilon_0 \cdot \left\{ n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^*$$

Combine this with the estimate (140) to obtain

$$(151) \quad \|\tilde{\omega}(\vec{v}_0, \dots, \vec{v}_n) - \vec{\omega}'(y_0, \dots, y_n)\| \leq 35\varepsilon_0 \cdot \left\{ n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^*$$

Then set

$$(152) \quad \omega(x_0, \dots, x_n) = \varphi_{\xi, \hat{u}}^g(\tilde{\omega}(\vec{v}_0, \dots, \vec{v}_n)), \quad r(x_0, \dots, x_n) = d_{\mathcal{F}}(x_0, \omega(x_0, \dots, x_n))$$

so we have $\{x_0, \dots, x_n\} \subset S_{\mathcal{F}}(\omega(x_0, \dots, x_n), r(x_0, \dots, x_n))$ as desired.

Recall that $\omega'(y_0, \dots, y_n) = \varphi_{\xi, \hat{u}}^g(\vec{\omega}'(y_0, \dots, y_n))$, then by Lemma 13.2 we have

$$(153) \quad d_{\mathcal{F}}(\omega(x_0, \dots, x_n), \omega'(y_0, \dots, y_n)) \leq \varepsilon_0 \cdot \left\{ 1 + 35n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^* < \varepsilon_3\lambda_{\mathcal{F}}^*/2$$

where the bound by $\varepsilon_3\lambda_{\mathcal{F}}^*/2$ follows from (91).

Finally, the estimate (135) follows from

$$\begin{aligned} |r(x_0, \dots, x_n) - r(y_0, \dots, y_n)| &= |d_{\mathcal{F}}(x_n, \omega(x_0, \dots, x_n)) - d_{\mathcal{F}}(y_n, \omega(y_0, \dots, y_n))| \\ &\leq |d_{\mathcal{F}}(x_n, \omega(x_0, \dots, x_n)) - d_{\mathcal{F}}(x_n, \omega'(y_0, \dots, y_n))| + 2\varepsilon_0\lambda_{\mathcal{F}}^* \\ &\leq |d_{\mathcal{F}}(\omega(x_0, \dots, x_n), \omega'(y_0, \dots, y_n))| + 2\varepsilon_0\lambda_{\mathcal{F}}^* \\ &\leq \varepsilon_3\lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0\lambda_{\mathcal{F}}^* < \varepsilon_3\lambda_{\mathcal{F}}^* \end{aligned}$$

This completes the proof of Proposition 15.2. \square

We have now established that for $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$, if $\{x_0, \dots, x_n\}$ is a transverse translate of the set $\{y_0, \dots, y_n\}$, then $\{x_0, \dots, x_n\}$ is $\tilde{\rho}_n > 3\varepsilon_2/2$ robust and admits an inscribed sphere whose radius varies according to the estimate (134). It remains to show that \mathcal{X} is stable.

PROPOSITION 15.5. *Let $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$. Assume given $1 \leq i_0 < i_1 < \dots < i_n \leq p_*$ and $\xi_{i_k} \in \Xi_{p_*} \subset L_0$ such that $y_k = \mathcal{X}_{i_k} \cap \mathcal{P}_{\theta_{i_n}}(y_n)$. Then for all $x_n \in \mathcal{X}_{i_n}$, $\Delta(x_0, \dots, x_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$.*

Proof. Let $\omega(y_0, \dots, y_n) \in \mathcal{P}_n(y_n)$ be the center of the inscribed sphere of radius $r(y_0, \dots, y_n)$. Then it is given that for all $\xi \in \mathcal{M}(y_n) - \{y_0, \dots, y_n\}$ we have that $d_{\mathcal{F}}(\xi, \omega(y_0, \dots, y_n)) > r(y_0, \dots, y_n)$.

We must show that the inscribed sphere for the set $\{x_0, \dots, x_n\}$ obtained in Proposition 15.2, with center $\omega(x_0, \dots, x_n)$ and radius $r(x_0, \dots, x_n)$, contains no points of \mathcal{X} in its interior. That is, we must show that

$$(154) \quad d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) > r(x_0, \dots, x_n) \quad \text{for all } \xi' \in \mathcal{M}(x_n) - \{x_0, \dots, x_n\}$$

Let $n \leq m \leq p_*$ be the largest m such that the condition (154) holds for all $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$ with $i_n \leq m$. If $m = p_*$ then we are done, so assume that $m < p_*$ and we show this leads to a contradiction. So we assume that we are given a simplex $\Delta(y_0, \dots, y_n)$ with $i_n = m + 1$, such that there is some $x_n \in \mathcal{X}_{i_n}$ and $\xi' \in \mathcal{M}(x_n) - \{x_0, \dots, x_n\}$ for which (154) fails.

First, consider the case where there exists $\xi' \in \mathcal{M}(x_n) - \{x_0, \dots, x_n\}$ such that

$$(155) \quad d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) \leq r(x_0, \dots, x_n) - 2\varepsilon_3\lambda_{\mathcal{F}}^*$$

Let $1 \leq q \leq p_*$ be such that $\xi' \in \mathcal{X}_q$ and set $\xi = \mathfrak{S}(\xi', \theta_{i_q}, \ell_0) \cap \mathcal{P}_n(y_n) \in \mathcal{M}(y_n)$. Then

$$\begin{aligned} d_{\mathcal{F}}(\xi, \omega(y_0, \dots, y_n)) &< d_{\mathcal{F}}(\xi', \omega'(y_0, \dots, y_n)) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* \quad \text{by Lemma 13.2} \\ &< d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 \quad \text{by (134)} \\ &< r(x_0, \dots, x_n) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \quad \text{by (155)} \\ &< r(y_0, \dots, y_n) - 3\varepsilon_3 \lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + (2\varepsilon_0 + \varepsilon_3/2) \lambda_{\mathcal{F}}^* \quad \text{by (135)} \\ &< r(y_0, \dots, y_n) + (2\varepsilon_0 - \varepsilon_3) \lambda_{\mathcal{F}}^* < r(y_0, \dots, y_n) \quad \text{by (87)} \end{aligned}$$

which contradicts the hypothesis that $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$. Thus, we may assume that

$$(156) \quad r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* < d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) \leq r(x_0, \dots, x_n)$$

Let $1 \leq q \leq p_*$ be the least such q such that there exists $x_n \in \mathcal{X}_{i_n}$ and (156) holds for $\xi' \in \mathcal{X}_q$. Note that $q \neq i_k$ for $0 \leq k \leq n$, and $\mathcal{X}_q = \mathfrak{S}(\xi_q, \theta_{i_q}, \ell_0)$ for some $\xi_q \in \Xi$, so that $\xi' = \mathfrak{S}(\xi_q, \theta_{i_q}, \ell_0) \cap \mathcal{P}_n(x_n)$.

We now use that ξ_q was chosen inductively to avoid the annular $2\varepsilon_1 \lambda_{\mathcal{F}}^*$ -thickening of the inscribed sphere for each n -simplex in $\Omega^{(n)}(\mathcal{M}(\xi_q))$.

First, we assume that $q > i_n$. For this subcase, we transfer the problem to the plaque $\mathcal{P}_n(\xi_q)$. Set $z_k = \mathcal{X}_{i_k} \cap \mathcal{P}_n(\xi_q)$ for $0 \leq k \leq n$. Note that $\{z_0, \dots, z_n\}$ admits an inscribed sphere by Proposition 15.2, with center $\omega(z_0, \dots, z_n)$ which satisfies

$$(157) \quad d_{\mathcal{F}}(\omega(z_0, \dots, z_n), \omega'(y_0, \dots, y_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

where $\omega'(y_0, \dots, y_n) = \phi_{i_n}(y_n, z_n)(\omega(y_0, \dots, y_n))$. Let $r(z_0, \dots, z_n)$ denote the radius of the sphere, which by (135) satisfies

$$(158) \quad r(y_0, \dots, y_n) - \varepsilon_3 \lambda_{\mathcal{F}}^* \leq r(z_0, \dots, z_n) \leq r(y_0, \dots, y_n) + \varepsilon_3 \lambda_{\mathcal{F}}^*$$

We claim that $\Delta(z_0, \dots, z_n) \in \Delta_{\mathcal{F}}^{(n)}(\widehat{\mathcal{X}}_{i_{q-1}})$. If not, then there exists $\eta' \in \widehat{\mathcal{X}}_{i_{q-1}} \cap \mathcal{P}_n(z_n)$ with $d_{\mathcal{F}}(\eta', \omega(z_0, \dots, z_n)) \leq r(z_0, \dots, z_n)$. This contradicts the minimality of the choice of q above. Thus, as ξ_q was chosen to satisfy the inequality (126), we have the estimate

$$(159) \quad d_{\mathcal{F}}(\xi_q, \omega(z_0, \dots, z_n)) > r(z_0, \dots, z_n) + 2\varepsilon_1 \lambda_{\mathcal{F}}^*$$

On the other hand, x_n was chosen so that for the inscribed sphere with center $\omega(x_0, \dots, x_n)$ and radius $r(x_0, \dots, x_n)$ we have the inequality (156) above.

Apply Proposition 15.2 to the cases $x_n \in \mathcal{X}_{i_n}$ and also $z_n \in \mathcal{X}_{i_n}$ to obtain the estimates (135) for both. Together, they imply

$$(160) \quad r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \leq r(z_0, \dots, z_n) \leq r(x_0, \dots, x_n) + 2\varepsilon_3 \lambda_{\mathcal{F}}^*$$

Also, (156) and (160) imply

$$(161) \quad d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \leq r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \leq r(z_0, \dots, z_n)$$

which for $\omega'(x_0, \dots, x_n) = \phi_{i_n}(x_n, z_n)(\omega(x_0, \dots, x_n))$ yields

$$(162) \quad d_{\mathcal{F}}(\xi_q, \omega'(x_0, \dots, x_n)) \leq r(z_0, \dots, z_n) + 2\varepsilon_3 \lambda_{\mathcal{F}}^* + \varepsilon_0 \lambda_{\mathcal{F}}^*$$

and thus by (157) and its corresponding version for x_n yields

$$\begin{aligned} d_{\mathcal{F}}(\xi_q, \omega(z_0, \dots, z_n)) &\leq d_{\mathcal{F}}(\xi_q, \omega'(x_0, \dots, x_n)) + d_{\mathcal{F}}(\omega'(x_0, \dots, x_n), \omega(z_0, \dots, z_n)) \\ &\leq (r(z_0, \dots, z_n) + (2\varepsilon_3 \lambda_{\mathcal{F}}^* + \varepsilon_0 \lambda_{\mathcal{F}}^*)) + (\varepsilon_3 \lambda_{\mathcal{F}}^* + 2\varepsilon_0 \lambda_{\mathcal{F}}^*) \\ (163) \quad &\leq r(z_0, \dots, z_n) + 4\varepsilon_3 \lambda_{\mathcal{F}}^* \end{aligned}$$

which by the choice $\varepsilon_3 < \varepsilon_1/2$ in (87) contradicts (159). Thus, the case $q > i_n$ is not possible.

Finally, consider the case where $q < i_n$. That is, the smallest q such that there exists $x_n \in \mathcal{X}_{i_n}$ and (156) holds for some $\xi' \in \mathcal{X}_q$ occurs for $q < i_n$. This means that in the process of constructing \mathcal{X} , we have chosen a point ξ_q which has distance greater than $2\varepsilon_1 \lambda_{\mathcal{F}}^*$ from all previously inscribed spheres for the net \mathcal{M}_{q-1} , but when we add the point ξ_{i_n} the Delaunay triangulation $\Delta_{\mathcal{F}}^{(n)}(\mathcal{M}_{i_n})$ abruptly changes on some leaves. The translates of ξ_{i_n} are contained both inside and outside of inscribed spheres, as the translates of ξ_q also wander inside and outside. We show this is impossible, due to

the choice of ε_1 and of the constants ε_3 and ε_4 which control how much the centers of inscribed spheres “wander” for transverse variation at most r_* .

Recall, we assume there is given $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$ and $x_n \in \mathcal{X}_{i_n}$ such that the transverse translate $\{x_0, \dots, x_n\}$ of the set $\{y_0, \dots, y_n\}$ is $\tilde{\rho}_n > 3\varepsilon_2/2$ robust. Thus by Proposition 15.2, there is an inscribed sphere with center $\omega(x_0, \dots, x_n)$ and radius $r(x_0, \dots, x_n)$ which satisfy

$$(164) \quad d_{\mathcal{F}}(\omega(x_0, \dots, x_n), \omega'(y_0, \dots, y_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

$$(165) \quad |r(x_0, \dots, x_n) - r(y_0, \dots, y_n)| < \varepsilon_3 \lambda_{\mathcal{F}}^*$$

where $\omega'(y_0, \dots, y_n) = \phi_{i_n}(y_n, x_n)(\omega(y_0, \dots, y_n))$ is the translate for the center of the inscribed sphere for the n -simplex $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$. There is also given $1 \leq q < i_n$ so that the translate $\xi' = \mathcal{X}_q \cap \mathcal{P}_n(x_n) \in \mathcal{M}(x_n)$ satisfies (156).

$$(166) \quad r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* < d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) \leq r(x_0, \dots, x_n)$$

Let $\{x'_0, \dots, x'_n\} = \{x_0, \dots, x_{n-1}, \xi'\}$ denote a reordering of the set so that $x'_k = \mathcal{X}_{i'_k} \cap \mathcal{P}_n(x_n)$ for $0 \leq k \leq n$ with $1 \leq i'_0 < \dots < i'_n \leq p_*$. Then these points satisfy, for $0 \leq k \leq n$,

$$(167) \quad r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \leq d_{\mathcal{F}}(x'_k, \omega(x_0, \dots, x_n)) \leq r(x_0, \dots, x_n)$$

The proof of Proposition 15.1 applied to the set $\{x'_0, \dots, x'_n\}$ yields that the collection is $\tilde{\rho}_n$ -robust, so admits an inscribed sphere by Proposition 15.2, with center $\omega(x'_0, \dots, x'_n)$ and radius $r(x'_0, \dots, x'_n)$. From the proof of Proposition 15.2, we have the estimates

$$(168) \quad d_{\mathcal{F}}(\omega(x'_0, \dots, x'_n), \omega(x_0, \dots, x_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

$$(169) \quad |r(x'_0, \dots, x'_n) - r(x_0, \dots, x_n)| < \varepsilon_3 \lambda_{\mathcal{F}}^*$$

Thus, combining (168) and (169), for $\zeta' = x_n$, we obtain

$$(170) \quad d_{\mathcal{F}}(\zeta', \omega(x'_0, \dots, x'_n)) \leq r(x'_0, \dots, x'_n) + 3\varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

Now let $\zeta = \xi_{i_n} \in \mathcal{X}_{i_n}$. The last step is to translate the points $\{x'_0, \dots, x'_n\}$ to the plaque $\mathcal{P}_n(\zeta)$, to obtain points $z'_k = \mathcal{X}_{i'_k} \cap \mathcal{P}_n(\zeta)$. Then $\{z'_0, \dots, z'_n\}$ is $\tilde{\rho}'_n$ -robust by Proposition 15.1, and admits an inscribed sphere with center $\omega(z'_0, \dots, z'_n)$ and radius $r(z'_0, \dots, z'_n)$ by Proposition 15.2. Moreover, this center and radius satisfy

$$(171) \quad d_{\mathcal{F}}(\omega(z'_0, \dots, z'_n), \omega'(x'_0, \dots, x'_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

$$(172) \quad |r(z'_0, \dots, z'_n) - r(x'_0, \dots, x'_n)| < \varepsilon_3 \lambda_{\mathcal{F}}^*$$

Combining (170), (171) and (172), we obtain

$$(173) \quad \begin{aligned} d_{\mathcal{F}}(\zeta, \omega(z'_0, \dots, z'_n)) &\leq d_{\mathcal{F}}(\zeta, \omega'(x'_0, \dots, x'_n)) + d_{\mathcal{F}}(\omega'(x'_0, \dots, x'_n), \omega(z'_0, \dots, z'_n)) \\ &\leq d_{\mathcal{F}}(\zeta', \omega(x'_0, \dots, x'_n)) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 \\ &\leq r(x'_0, \dots, x'_n) + 3\varepsilon_3 \lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 \\ &\leq r(z'_0, \dots, z'_n) + \varepsilon_3 \lambda_{\mathcal{F}}^* + 3\varepsilon_3 \lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 \\ &< r(z'_0, \dots, z'_n) + 4\varepsilon_3 \lambda_{\mathcal{F}}^* + 2\varepsilon_0 \lambda_{\mathcal{F}}^* \end{aligned}$$

By the choice of ε_3 in (87) we have $4\varepsilon_3 + 2\varepsilon_0 < 2\varepsilon_1$, so that (173) contradicts the choice of $\zeta = \xi_{i_n}$ to satisfy

$$d_{\mathcal{F}}(\zeta, \omega(z'_0, \dots, z'_n)) \geq r(z'_0, \dots, z'_n) + 2\varepsilon_1 \lambda_{\mathcal{F}}^*$$

Thus, the case $q < i_n$ again leads to a contraction. This completes the proof of Proposition 15.5. \square

16. BIG BOX LEMMA

We recall the statement that we prove. Let $L_0 \subset \mathfrak{M}$ be a leaf, and $\tilde{K} \subset \tilde{L}$ a connected compact subset of the holonomy covering $\Pi: \tilde{L}_0 \rightarrow L_0$, and set $K = \Pi(\tilde{K})$. We assume that \tilde{K} is a union of plaques for convenience. Assume also that the restriction of the covering map, $\Pi: \tilde{K} \rightarrow L_0$, is an injection. It follows that for a basepoint $x_0 \in \mathcal{N}_K \equiv K \cap \mathcal{N}_0$, there are no closed paths with non-trivial holonomy based at x_0 . Thus, we can identify the points \mathcal{N}_K with the finite set $\tilde{K} \cap \tilde{\mathcal{N}}_0$ as in Section 14. In particular, \mathcal{N}_K is finite. Define the constant

$$(174) \quad \epsilon'_K = \min\{d_{\mathcal{F}}(x, y) \mid x \neq y, x, y \in \mathcal{N}_K\}$$

Let $\epsilon_K = \min\{\epsilon'_K, r_*/2\}$ where r_* is as in Section 14.

Then by Proposition 4.9, and using that \mathcal{N}_K is a finite set, there exists $\delta_K > 0$ such that

$$(175) \quad h_x(D_{\mathfrak{X}}(x_0, \delta_K)) \subset D_{\mathfrak{X}}(x, \epsilon_K/3) \quad , \quad \forall x \in \mathcal{N}_K$$

Choose a clopen set $x_0 \in V \subset D_{\mathfrak{X}}(x_0, \delta_K)$, then the set of images

$$\{V_x \equiv h_x(V) \mid x \in \mathcal{N}_K\}$$

are disjoint. We can then form the Reeb neighborhood $\mathfrak{N}(K, V)$ of K as defined by (22).

We are now in the situation of Section 14, except that instead of constructing the sets Λ_p and Ξ_p in the full leaf L_0 we restrict the procedure to the subset $x_0 \in K \subset L_0$. The inductive procedure stops when $K \subset \text{Pen}_{\mathcal{F}}(\mathcal{M}_p, d_2'')$.

Finally, suppose that we are given a union of standard sections $\mathcal{X}' \subset \mathfrak{N}(K, V)$ so that there intersection with K satisfies the inductive hypotheses in Section 14, then instead of starting with $\Lambda_1 = \{x_0\}$ we take $\Lambda'_1 = \mathcal{X}' \cap K$. Then proceed as in the equicontinuous case. This completes the proof of Theorem 1.2.

The local Reeb product we have hereby established is an essential tool in the study of general matchbox manifolds and has application beyond the equicontinuous setting.

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VORONOI TESSELLATIONS FOR MATCHBOX MANIFOLDS

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ABSTRACT. Matchbox manifolds \mathfrak{M} are a special class of foliated spaces, which include as special examples the weak solenoids, suspensions of odometer and Toeplitz actions, and tiling spaces associated to aperiodic tilings with finite local complexity. They have many properties analogous to those of a compact manifold, but the additional data inherently encoded in the pseudogroup dynamics of its foliation \mathcal{F} represent fundamental groupoid data. As such, they are a rich class of mathematical objects to study. The special cases of weak solenoids, suspensions of group actions, and tiling spaces have an additional structure, that of a transverse foliation, consisting of a continuous family of Cantor sets transverse to the foliated structure. The purpose of this paper is to show that this transverse structure can be defined on all equicontinuous matchbox manifolds, as well as on special foliated subsets of general matchbox manifolds. This follows from the construction of uniform Voronoi tessellations on leaves, which is the main goal of this work. From this, we define a foliated Delaunay triangulation of \mathfrak{M} , adapted to the dynamics of \mathcal{F} . The result is highly technical, but underlies the study of the basic topological structure of matchbox manifolds in general. Our methods are similar to some prior results in the literature [35, 43], though are unique in that we give the construction of the Voronoi tessellations for a complete Riemannian manifold L of arbitrary dimension, while the constructions in the literature apply only to the case where the manifold L is Euclidean space \mathbb{R}^n .

1. INTRODUCTION

Dynamical systems theory on a smooth closed manifold M studies the orbits of a non-singular flow on M , or the orbits of a diffeomorphism of M . This notion can be extended to the study of dynamical systems defined by the leaves of a foliation \mathcal{F} of M , or the orbits of a finitely-generated group of diffeomorphisms of M , a generalized interpretation of dynamical systems promoted in the seminal paper of Smale [58]. The area of foliation dynamics is discussed in the recent lecture notes [38].

The study of the dynamics of a topological action of finitely generated group Γ on a Cantor set K lies at a somewhat opposite extreme from smooth dynamical systems. The traditional aspects of this theory encompasses a wide range of examples, from the study of odometers and Toeplitz flows [24, 26], to the study of hyperbolic automorphisms of “Smale spaces” as in the works [40, 49, 50], and minimal actions of \mathbb{Z}^n on Cantor sets as studied in [34, 35] and many other works.

A *matchbox manifold* is a foliated space \mathfrak{M} , which is a continuum equipped with a local product structure whose local transverse models are totally disconnected, and leaves with a smooth structure. This terminology was introduced for one-dimensional continua in [1, 3, 4], and is based on the intuition that a 1-dimensional matchbox manifold \mathfrak{M} has local coordinate charts U which are homeomorphic to a “box of matches”; for $n > 1$ the “matches” are n -dimensional plaques. Definition 2.4 below of a matchbox manifold makes precise various technical conditions imposed.

The study of matchbox manifolds is intermediate between these two classes of examples of dynamical systems, as it assumes there are smooth leaves, but the transverse geometry is zero-dimensional. When the leaves have dimension at least two, the geometry of the leaves become a key aspect of the study of their dynamics, so the class of examples is broader than the more well-studied case of \mathbb{Z}^n -actions, which corresponds to foliations with Euclidean leaves.

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The classical solenoids modeled on \mathbb{S}^1 are 1-dimensional matchbox manifolds, and generalized solenoids with base an n -dimensional manifold, as in [30, 44, 57], are examples of n -dimensional matchbox manifolds. The tiling spaces associated to aperiodic tilings of \mathbb{R}^n (or other homogeneous spaces) with finite local complexity provide another large class of examples [7, 12]. A variety of further examples can be found in [17, Chapter 11] and the works [14, 15, 21, 22, 12, 33, 45].

For the special cases where \mathfrak{M} is a weak solenoid or a tiling space, there is a natural inverse limit structure for \mathfrak{M} with a compact initial base space M . The base M is a manifold in the case that \mathfrak{M} is a weak solenoid, and M is a branched manifold in the case that \mathfrak{M} is a tiling space. In both cases, there is given a continuous family of local transversals to the foliation \mathcal{F} on \mathfrak{M} . The collection of these transversals define a *transverse Cantor foliation* \mathcal{H} of \mathfrak{M} , whose “leaves” are themselves Cantor sets. This concept is made precise in Section 6. In this paper we study the question, does every matchbox manifold \mathfrak{M} admit such a transverse Cantor foliation \mathcal{H} ?

The importance of having a global bi-foliated structure on \mathfrak{M} , in the form of local coordinate charts which define the manifold leaves of \mathcal{F} and the Cantor set leaves of \mathcal{H} , is that it allows finite approximations of the dynamical structure of \mathcal{F} , acting on the transverse leaves of \mathcal{H} by the holonomy groupoid [7, 12, 20, 21]. As a consequence, the authors in [20, 21, 22] extend to the generality of matchbox manifolds many of the results concerning the classification of tilings spaces as in [9, 10, 11, 32, 39, 55, 56].

The existence and properties of a transverse Cantor foliation \mathcal{H} for a matchbox manifold \mathfrak{M} depends on the dynamical properties of \mathcal{F} . Our first result gives a solution for equicontinuous dynamics.

THEOREM 1.1. *Let \mathfrak{M} be an equicontinuous matchbox manifold. Then there exists a transverse Cantor foliation \mathcal{H} on \mathfrak{M} such that the projection to the leaf space $\mathfrak{M} \rightarrow \mathfrak{M}/\mathcal{H} \cong M$ is a Cantor bundle map over a compact manifold M .*

The same techniques behind the proof of Theorem 1.1 also yields a more basic result, which has a variety of applications for the study of general matchbox manifolds.

The “Long Box Lemma” is a fundamental result for the study of non-singular flows on continua, and appears explicitly in works of Aarts, Fokkink and Oversteegen [29, Lemma 5.2], and also [1, 2, 3, 30]. The lemma states that every *connected, contractible orbit segment* K in a 1-dimensional matchbox manifold is contained in a bi-foliated open subset \mathfrak{N}_K of \mathfrak{M} , which is the “long box” neighborhood of K . We generalize this result to n -dimensional matchbox manifolds, for $n \geq 1$.

DEFINITION 1.2. *Let \mathfrak{M} be a matchbox manifold, $x \in \mathfrak{M}$ a basepoint, $L_x \subset \mathfrak{M}$ the leaf through x , and \tilde{L}_x the holonomy covering of L_x . We say that $K_x \subset L_x$ is a *proper base* if K_x is a union of closed foliation plaques with $x \in K_x$, and there is a connected compact subset $\tilde{K}_x \subset \tilde{L}_x$ such that the composition $\iota_x: \tilde{K}_x \subset \tilde{L}_x \rightarrow L_x \subset \mathfrak{M}$ is injective with image K_x .*

Note that K_x is therefore path connected, and the holonomy of \mathcal{F} along any path in K_x is trivial. The set K_x need not be simply connected, though. Our second result is foundational, as it implies the existence of a bi-foliated neighborhood of K_x , the “Big Box”.

THEOREM 1.3 (Big Box). *Let \mathfrak{M} be a matchbox manifold, $x \in \mathfrak{M}$ and $K_x \subset L_x \subset \mathfrak{M}$ a proper base. Then there exists a clopen transversal $V_x \subset \mathfrak{M}$ containing x , and a foliated homeomorphic inclusion $\Phi: K_x \times V_x \rightarrow \mathfrak{N}_{K_x} \subset \mathfrak{M}$ such that the images $\Phi(\{y\} \times V_x \mid y \in K_x)$ form a continuous family of Cantor transversals for $\mathcal{F}|_{\mathfrak{N}_{K_x}}$ in the sense of Definition 6.1.*

In the case of one-dimensional matchbox manifolds, Theorem 1.3 reduces to the Long Box Lemma, as a contractible line segment K has no holonomy. The requirement that K_x contains no loops with holonomy is essential.

Theorem 1.3, and especially the methods used in its proof, have a variety of applications, analogous to what has been shown for the special case of tiling spaces. Theorems 1.1 and 1.3 underlie the authors’ studies of matchbox manifolds in [20, 21, 22], and our further program of research on their properties and classification.

A notable aspect of the study of matchbox manifolds, is that because the ambient space is only assumed to be a continuum, that is, a compact, connected, and non-empty metrizable space, many of the standard techniques of manifolds and their dynamics are not available. This makes their study much more technical at times, as evidenced by the proofs in this paper which develops a foundation for their general study.

For example, for a matchbox manifold \mathfrak{M} , there is no transverse exponential map, as its transversal geometry is totally disconnected. For a smooth foliation of a closed manifold M , if we consider the smooth analog of Theorem 1.3, where K_x is a compact subset without holonomy of a leaf, then the analogous assertion is simply a version of the Reeb Stability Theorem, and the existence of a local product structure follows almost trivially from the existence of a transverse exponential map to the leaves. However, in our context, the conclusions of Theorems 1.1 and 1.3 are obtained only with significant technical effort.

Our techniques are based on “elementary methods”, but are quite technical and involved due to a number of factors. For each foliated coordinate chart, $\overline{U}_i \subset \mathfrak{M}$, there is a natural “vertical” foliation whose leaves are the images of the vertical transversals defined by the coordinate charts. The problem is that on the overlap of two charts, these vertical foliations need not match up, as the only requirement on a foliation chart for \mathcal{F} is that the horizontal plaques in each chart “match up”. The exception is when \mathfrak{M} is given with a fibration structure, then the coordinates can be chosen to be adapted to the fibration structure, and so the fibers of the bundle restrict to transversals in each chart which are compatible on overlapping charts.

The idea of our construction is to subdivide the topological space \mathfrak{M} by dividing it into “arbitrarily small” coordinate boxes, where the problems to be solved become “almost linear”. It is also curious to note that the philosophy of our approach is reminiscent of the original technique by Sullivan to define the polynomial model for smooth manifolds in rational homotopy theory [36, 59], where topological data is linearized in this way.

One notable aspect of this work, is that we develop the theory of uniform triangulations of the leaves of a matchbox manifold \mathfrak{M} , such that the triangles have sufficiently small diameter and are in “general position”, so that they are stable in transverse directions for small perturbations. A uniform triangulation of the leaves, satisfying the required stability conditions, is constructed as the *Delaunay simplicial complex* associated to a *Voronoi tessellation* of the leaves.

The leafwise net defining this Voronoi tessellation is chosen to have sufficiently small spacing, and satisfy estimates imposed by the requirements of general position for the associated triangulation. The proof that all this can be done is quite tedious. However, we believe the exposition and careful development of this method makes a useful contribution to the theory of Voronoi tessellations on general manifolds, and that the details presented are necessary as they do not seem to be well-known.

One reason for the technicalities encountered, is that we give effective estimates for the construction of Delaunay triangulations in the case where the leaves of \mathcal{F} are general Riemannian manifolds, and are not assumed to be Euclidean as in [43, 35] for example. Another reason, and more unexpected, is that the construction of “stable” Delaunay triangulations, as required by the foliation product structure, is fundamentally more subtle in dimensions greater than two. This leads to the most technically detailed aspects of this work. The authors present this construction in full detail, as it does not seem to be dealt with in the literature, yet is the foundation for a variety of other results. We note that our methods are related to the methods used by T. Giordano, H. Matui, I. Putnam, and C. Skau in their study of affability for \mathbb{Z}^d -Cantor minimal systems [35, 43], which develop a theory of Delaunay triangulations for higher dimensional Euclidean spaces. The methods of this paper are more geometric and so apply more generally, and hence are of independent interest.

The structure of the paper is as follows. In Part I, we present the basic concepts and dynamical properties of matchbox manifolds as required. The proofs of results in Part 1, Sections 2 through Section 5, are often omitted, or when details of proof or notation are necessary for later development, they are briefly outlined. Full proofs of all of these dynamical results are given in [20].

Section 2 gives definitions and notations. Then Section 3 introduces the holonomy pseudogroup, and gives some basic technical properties of holonomy maps. Section 4 recalls important classical definitions from topological dynamics, adapted to the case of matchbox manifolds, and gives several results concerning the dynamical properties of matchbox manifolds. The notion of a *transverse Cantor foliation* \mathcal{H} of \mathfrak{M} is defined in Section 6. Finally, we conclude Part I of the paper with a discussion of the *Reeb Stability Theorem* for matchbox manifolds in Section 5, which is a basic concept in any study of foliation dynamics.

In Part II, Sections 7 and 8 give the classical constructions of Voronoi and Delaunay triangulations in the context of Riemannian manifolds. In Section 9, we extend these concepts from a single leaf, to a “parametrized version” which applies uniformly to the leaves of a matchbox manifold \mathfrak{M} . Then in Section 6, we introduce the notion of *transverse Cantor foliations* for matchbox manifolds, and in Section 10 show how to obtain a transverse Cantor foliation from a nice stable transversal.

In Part III, Sections 11 and 12 consider Euclidean Voronoi and Delaunay triangulations and the manipulations of these that are required in the subsequent constructions in the Riemannian context.

In Part IV, Sections 13 and 15 are the most technically demanding sections. The adaptation of the Voronoi tessellations, from a flat Euclidean structure to general Riemannian manifolds, requires the introduction of many local estimates on the metric geometry.

Finally, in Part V, Sections 16 and 17 give the inductive construction for nice stable transversals, and Section 18 completes the proof of their stability properties. Then in the last Section 19 we apply these results to complete the proofs of Theorems 1.1 and 1.3.

PART I - MATCHBOX MANIFOLD DYNAMICS

In this part, we discuss the basic concepts of foliated spaces. Further discussion with examples can be found in [17, Chapter 11], [47, Chapter 2] and the first two authors’ papers [19, 20]. We give here precise definitions and formulate some of their basic geometric and dynamical properties. When appropriate, the reader is referred to the paper [20] for proofs, as there is significant overlap between the material in Part I, especially Sections 2 and 3, and the development given there.

2. FOLIATED SPACES

DEFINITION 2.1. A foliated space of dimension n is a continuum \mathfrak{M} , such that there exists a compact separable metric space \mathfrak{X} , and for each $x \in \mathfrak{M}$ there is a compact subset $\mathfrak{T}_x \subset \mathfrak{X}$, an open subset $U_x \subset \mathfrak{M}$, and a homeomorphism defined on the closure $\varphi_x: \overline{U}_x \rightarrow [-1, 1]^n \times \mathfrak{T}_x$ such that $\varphi_x(x) = (0, w_x)$ where $w_x \in \text{int}(\mathfrak{T}_x)$. The subspace \mathfrak{T}_x of \mathfrak{X} is the local transverse model at x .

Note that the assumption that φ_x is defined on closed sets $[-1, 1]^n$ implies that it admits an extension to a foliation chart $\widehat{\varphi}_x: (-2, 2)^n \times \mathfrak{T}_x \rightarrow \widehat{U}_x$ where $\overline{U}_x \subset \widehat{U}_x$.

Let $\pi_x: \overline{U}_x \rightarrow \mathfrak{T}_x$ denote the composition of φ_x with projection onto the second factor.

For $w \in \mathfrak{T}_x$ the set $\mathcal{P}_x(w) = \pi_x^{-1}(w) \subset \overline{U}_x$ is called a *plaque* for the coordinate chart φ_x . We adopt the notation, for $z \in \overline{U}_x$, that $\mathcal{P}_x(z) = \mathcal{P}_x(\pi_x(z))$, so that $z \in \mathcal{P}_x(z)$. Note that each plaque $\mathcal{P}_x(w)$ is given the topology so that the restriction $\varphi_x: \mathcal{P}_x(w) \rightarrow [-1, 1]^n \times \{w\}$ is a homeomorphism. Then $\text{int}(\mathcal{P}_x(w)) = \varphi_x^{-1}((-1, 1)^n \times \{w\})$.

Let $U_x = \text{int}(\overline{U}_x) = \varphi_x^{-1}((-1, 1)^n \times \text{int}(\mathfrak{T}_x))$. Note that if $z \in U_x \cap U_y$, then $\text{int}(\mathcal{P}_x(z)) \cap \text{int}(\mathcal{P}_y(z))$ is an open subset of both $\mathcal{P}_x(z)$ and $\mathcal{P}_y(z)$. The collection of sets

$$\mathcal{V} = \{\varphi_x^{-1}(V \times \{w\}) \mid x \in \mathfrak{M}, w \in \mathfrak{T}_x, V \subset (-1, 1)^n \text{ open}\}$$

forms the basis for the *fine topology* of \mathfrak{M} . The connected components of the fine topology are called leaves, and define the foliation \mathcal{F} of \mathfrak{M} . For $x \in \mathfrak{M}$, let $L_x \subset \mathfrak{M}$ denote the leaf of \mathcal{F} containing x .

Note that in Definition 2.1, the collection of transverse models $\{\mathfrak{T}_x \mid x \in \mathfrak{M}\}$ need not have union equal to \mathfrak{X} . This is similar to the situation for a smooth foliation of codimension q , where each foliation chart projects to an open subset of \mathbb{R}^q , but the collection of images need not cover \mathbb{R}^q .

DEFINITION 2.2. A smooth foliated space is a foliated space \mathfrak{M} as above, such that there exists a choice of local charts $\varphi_x: \overline{U}_x \rightarrow [-1, 1]^n \times \mathfrak{T}_x$ such that for all $x, y \in \mathfrak{M}$ with $z \in U_x \cap U_y$, there exists an open set $z \in V_z \subset U_x \cap U_y$ such that $\mathcal{P}_x(z) \cap V_z$ and $\mathcal{P}_y(z) \cap V_z$ are connected open sets, and the composition

$$\psi_{x,y;z} \equiv \varphi_y \circ \varphi_x^{-1}: \varphi_x(\mathcal{P}_x(z) \cap V_z) \rightarrow \varphi_y(\mathcal{P}_y(z) \cap V_z)$$

is a smooth map, where $\varphi_x(\mathcal{P}_x(z) \cap V_z) \subset \mathbb{R}^n \times \{w\} \cong \mathbb{R}^n$ and $\varphi_y(\mathcal{P}_y(z) \cap V_z) \subset \mathbb{R}^n \times \{w'\} \cong \mathbb{R}^n$. The leafwise transition maps $\psi_{x,y;z}$ are assumed to depend continuously on z in the C^∞ -topology on maps between subsets of \mathbb{R}^n .

A map $f: \mathfrak{M} \rightarrow \mathbb{R}$ is said to be *smooth* if for each flow box $\varphi_x: \overline{U}_x \rightarrow [-1, 1]^n \times \mathfrak{T}_x$ and $w \in \mathfrak{T}_x$ the composition $y \mapsto f \circ \varphi_x^{-1}(y, w)$ is a smooth function of $y \in (-1, 1)^n$, and depends continuously on w in the C^∞ -topology on maps of the plaque coordinates y . As noted in [47] and [17, Chapter 11], this allows one to define smooth partitions of unity, vector bundles, and tensors for smooth foliated spaces. In particular, one can define leafwise Riemannian metrics. We recall a standard result, whose proof for foliated spaces can be found in [17, Theorem 11.4.3].

THEOREM 2.3. Let \mathfrak{M} be a smooth foliated space. Then there exists a leafwise Riemannian metric for \mathcal{F} , such that for each $x \in \mathfrak{M}$, L_x inherits the structure of a complete Riemannian manifold with bounded geometry, and the Riemannian geometry depends continuously on x . \square

Bounded geometry implies, for example, that for each $x \in \mathfrak{M}$, there is a leafwise exponential map $\exp_x^{\mathcal{F}}: T_x \mathcal{F} \rightarrow L_x$ which is a surjection, and the composition $\exp_x^{\mathcal{F}}: T_x \mathcal{F} \rightarrow L_x \subset \mathfrak{M}$ depends continuously on x in the compact-open topology on maps.

DEFINITION 2.4. A matchbox manifold is a continuum with the structure of a smooth foliated space \mathfrak{M} , such that the transverse model space \mathfrak{X} is totally disconnected, and for each $x \in \mathfrak{M}$, $\mathfrak{T}_x \subset \mathfrak{X}$ is a clopen subset.

2.1. Metric properties and regular covers. For the rest of this paper, all foliated spaces are assumed to be smooth with a given leafwise Riemannian metric. The study of the dynamics of a foliated space \mathfrak{M} requires generalizing various concepts for flows (and more generally group actions) from the orbits of points to the properties of leaves L in a foliated space. On a technical level, it is very useful in developing these generalizations to have a *strong local convexity property* for the leaves, generalizing the local convexity of the orbit of a flow.

Another nuance about the definition of foliated spaces, and matchbox manifolds in particular, is that for given $x \in \mathfrak{M}$, the neighborhood \overline{U}_x in Definition 2.1 need not be “local”. As the transversal model \mathfrak{T}_x need not be connected, the set \overline{U}_x need not be connected, and *a priori* its connected components need not be contained in a suitably small metric ball around x . The technical procedures described in detail in [20, Section 2.1 - 2.2] ensure that we can always choose local charts for \mathfrak{M} to have a *uniform locality property*, as well as other metric regularity properties as discussed below.

Let $d_{\mathfrak{M}}: \mathfrak{M} \times \mathfrak{M} \rightarrow [0, \infty)$ denote the metric on \mathfrak{M} , and $d_{\mathfrak{X}}: \mathfrak{X} \times \mathfrak{X} \rightarrow [0, \infty)$ the metric on \mathfrak{X} .

For $x \in \mathfrak{M}$ and $\epsilon > 0$, let $D_{\mathfrak{M}}(x, \epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x, y) \leq \epsilon\}$ be the closed ϵ -ball about x in \mathfrak{M} , and $B_{\mathfrak{M}}(x, \epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x, y) < \epsilon\}$ the open ϵ -ball about x .

Similarly, for $w \in \mathfrak{X}$ and $\epsilon > 0$, let $D_{\mathfrak{X}}(w, \epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w, w') \leq \epsilon\}$ be the closed ϵ -ball about w in \mathfrak{X} , and $B_{\mathfrak{X}}(w, \epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w, w') < \epsilon\}$ the open ϵ -ball about w .

Each leaf $L \subset \mathfrak{M}$ has a complete path-length metric, induced from the leafwise Riemannian metric:

$$d_{\mathcal{F}}(x, y) = \inf \{ \|\gamma\| \mid \gamma: [0, 1] \rightarrow L \text{ is piecewise } C^1, \gamma(0) = x, \gamma(1) = y, \gamma(t) \in L \quad \forall 0 \leq t \leq 1 \}$$

and where $\|\gamma\|$ denotes the path length of the piecewise C^1 -curve $\gamma(t)$. If $x, y \in \mathfrak{M}$ are not on the same leaf, then set $d_{\mathcal{F}}(x, y) = \infty$. For each $x \in \mathfrak{M}$ and $r > 0$, let $D_{\mathcal{F}}(x, r) = \{y \in L_x \mid d_{\mathcal{F}}(x, y) \leq r\}$.

Note that the metric $d_{\mathfrak{M}}$ on \mathfrak{M} and the leafwise metric $d_{\mathcal{F}}$ have no relation, beyond their relative continuity properties. The metric $d_{\mathfrak{M}}$ is essentially just used to define the metric topology on \mathfrak{M} , while the metric $d_{\mathcal{F}}$ depends on an independent choice of the Riemannian metric on leaves.

For each $x \in \mathfrak{M}$, the Gauss Lemma implies that there exists $\lambda_x > 0$ such that $D_{\mathcal{F}}(x, \lambda_x)$ is a *strongly convex* subset for the metric $d_{\mathcal{F}}$. That is, for any pair of points $y, y' \in D_{\mathcal{F}}(x, \lambda_x)$ there is a unique shortest geodesic segment in L_x joining y and y' and it is contained in $D_{\mathcal{F}}(x, \lambda_x)$ (cf. [25, Chapter 3, Proposition 4.2]). Note then, that for all $0 < \lambda < \lambda_x$ the disk $D_{\mathcal{F}}(x, \lambda)$ is also strongly convex. Then using that \mathfrak{M} is compact and the leafwise metrics have uniformly bounded geometry, we obtain:

LEMMA 2.5. *There exists $\lambda_{\mathcal{F}} > 0$ such that for all $x \in \mathfrak{M}$, $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}})$ is strongly convex.*

If \mathcal{F} is defined by a flow without periodic points, so that every leaf is diffeomorphic to \mathbb{R} , then the entire leaf is strongly convex, so $\lambda_{\mathcal{F}} > 0$ can be chosen arbitrarily. For foliations with leaves of dimension $n > 1$, the constant $\lambda_{\mathcal{F}}$ must be less than the injectivity radius for each of the leaves.

The following proposition summarizes results in [20, Sections 2.1 - 2.2].

PROPOSITION 2.6. [20] *For a smooth foliated space \mathfrak{M} , there exist constants $\epsilon_{\mathfrak{M}}, \lambda_{\mathcal{F}} > 0$ and a choice of local charts $\varphi_x: \overline{U}_x \rightarrow [-1, 1]^n \times \mathfrak{T}_x$ with the following properties:*

- (1) *For each $x \in \mathfrak{M}$, $U_x \equiv \text{int}(\overline{U}_x) = \varphi_x^{-1}((-1, 1)^n \times B_{\mathfrak{X}}(w_x, \epsilon_x))$, where $\epsilon_x > 0$.*
- (2) *Locality: for all $x \in \mathfrak{M}$ each $\overline{U}_x \subset B_{\mathfrak{M}}(x, \epsilon_{\mathfrak{M}})$.*
- (3) *Local convexity: for all $x \in \mathfrak{M}$ the plaques of φ_x are leafwise strongly convex subsets with diameter less than $\lambda_{\mathcal{F}}/2$. That is, there is a unique shortest geodesic segment joining any two points in a plaque, and the entire geodesic segment is contained in the plaque.*

A *regular covering* of \mathfrak{M} is one that satisfies the conditions of Proposition 2.6.

By a standard argument, there exists a finite collection $\{x_1, \dots, x_{\nu}\} \subset \mathfrak{M}$ where $\varphi_{x_i}(x_i) = (0, w_{x_i})$ for $w_{x_i} \in \mathfrak{X}$, and regular foliation charts $\varphi_{x_i}: \overline{U}_{x_i} \rightarrow [-1, 1]^n \times \mathfrak{T}_{x_i}$ satisfying the conditions of Proposition 2.6, which form an open covering of \mathfrak{M} . Moreover, without loss of generality, we can impose a uniform size restriction on the plaques of each chart. Without loss of generality, we can assume there exists $0 < \delta_{\mathcal{U}}^{\mathcal{F}} < \lambda_{\mathcal{F}}/4$ so that for all $1 \leq i \leq \nu$ and $\omega \in \mathfrak{T}_i$ with $x_{\omega} = \varphi_{x_i}^{-1}(0, \omega)$, then the plaque for φ_{x_i} through x_{ω} satisfies the uniform estimate of diameters:

$$(1) \quad D_{\mathcal{F}}(x_{\omega}, \delta_{\mathcal{U}}^{\mathcal{F}}/2) \subset \mathcal{P}_i(\omega) \subset D_{\mathcal{F}}(x_{\omega}, \delta_{\mathcal{U}}^{\mathcal{F}})$$

For each $1 \leq i \leq \nu$ the set $\mathcal{T}_{x_i} = \varphi_i^{-1}(0, \mathfrak{T}_i)$ is a compact transversal to \mathcal{F} . Again, without loss of generality, we can assume that the transversals $\{\mathcal{T}_{x_1}, \dots, \mathcal{T}_{x_{\nu}}\}$ are pairwise disjoint, so there exists a constant $0 < e_1 < \delta_{\mathcal{U}}^{\mathcal{F}}$ such that

$$(2) \quad d_{\mathcal{F}}(x, y) \geq e_1 \quad \text{for } x \neq y, x \in \mathcal{T}_{x_i}, y \in \mathcal{T}_{x_j}, 1 \leq i, j \leq \nu$$

In particular, this implies that the centers of disjoint plaques on the same leaf are separated by distance at least e_1 .

Given a fixed choice of foliation covering as above, we simplify the notation as follows. For $1 \leq i \leq \nu$, set $\overline{U}_i = \overline{U}_{x_i}$, $U_i = U_{x_i}$, and $\epsilon_i = \epsilon_{x_i}$. Let $\mathcal{U} = \{U_1, \dots, U_{\nu}\}$ denote the corresponding open covering of \mathfrak{M} , with coordinate maps

$$\varphi_i = \varphi_{x_i}: \overline{U}_i \rightarrow [-1, 1]^n \times \mathfrak{T}_i, \quad \pi_i = \pi_{x_i}: \overline{U}_i \rightarrow \mathfrak{T}_i, \quad \lambda_i: \overline{U}_i \rightarrow [-1, 1]^n.$$

For $z \in \overline{U}_i$, the plaque of the chart φ_i through z is denoted by $\mathcal{P}_i(z) = \mathcal{P}_i(\pi_i(z)) \subset \overline{U}_i$. Note that the restriction $\lambda_i: \mathcal{P}_i(z) \rightarrow [-1, 1]^n$ is a homeomorphism onto. Also, define sections

$$\tau_i: \mathfrak{T}_i \rightarrow \overline{U}_i, \text{ defined by } \tau_i(\xi) = \varphi_i^{-1}(0, \xi), \text{ so that } \pi_i(\tau_i(\xi)) = \xi.$$

Then $\mathcal{T}_i = \mathcal{T}_{x_i}$ is the image of τ_i and we let $\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_{\nu} \subset \mathfrak{M}$ denote their disjoint union.

Let $\mathfrak{T}_* = \mathfrak{T}_1 \cup \dots \cup \mathfrak{T}_{\nu} \subset \mathfrak{X}$; note that \mathfrak{T}_* is compact, and if each \mathfrak{T}_i is totally disconnected, then \mathfrak{T}_* will also be totally disconnected.

We assume in the following that a finite regular covering \mathcal{U} of \mathfrak{M} as above has been chosen.

2.2. Foliated maps. A *leafwise path*, or more precisely an \mathcal{F} -*path*, is a continuous map $\gamma: [0, 1] \rightarrow \mathfrak{M}$ such that there is a leaf L of \mathcal{F} for which $\gamma(t) \in L$ for all $0 \leq t \leq 1$.

If \mathfrak{M} is a matchbox manifold, and $\gamma: [0, 1] \rightarrow \mathfrak{M}$ is continuous, then the standard arguments, relying on the fact that the composition of γ with charts maps must be continuous, and the transversals \mathfrak{T}_i are totally disconnected, show that $\gamma(t)$ is a leafwise path. More generally, this argument shows that the continuous image of any path-connected space X must be contained in a single leaf of \mathfrak{M} , and the following fundamental lemma.

LEMMA 2.7. *Let \mathfrak{M} and \mathfrak{M}' be matchbox manifolds, and $h: \mathfrak{M}' \rightarrow \mathfrak{M}$ a continuous map. Then h maps the leaves of \mathcal{F}' to leaves of \mathcal{F} . In particular, any homeomorphism $h: \mathfrak{M} \rightarrow \mathfrak{M}$ of a matchbox manifold is a foliated map.*

2.3. Local estimates. We next introduce a number of constants based on the above choices, which will be used throughout the paper when making metric estimates.

Let $\epsilon_{\mathcal{U}} > 0$ be a Lebesgue number for the covering \mathcal{U} . That is, given any $z \in \mathfrak{M}$ there exists some index $1 \leq i_z \leq \nu$ such that the open metric ball $B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$.

The local projections $\pi_i: \overline{U}_i \rightarrow \mathfrak{T}_i$ and sections $\tau_i: \mathfrak{T}_i \rightarrow \overline{U}_i$ are continuous maps of compact spaces, so admit uniform metric estimates as follows.

LEMMA 2.8. [20] *There exists a continuous increasing function ρ_{π} (the modulus of continuity for the projections π_i) such that:*

$$(3) \quad \forall 1 \leq i \leq \nu \text{ and } x, y \in \overline{U}_i \quad , \quad d_{\mathfrak{M}}(x, y) < \rho_{\pi}(\epsilon) \implies d_{\mathfrak{X}}(\pi_i(x), \pi_i(y)) < \epsilon \text{ .}$$

Proof. Set $\rho_{\pi}(\epsilon) = \min \{ \epsilon, \min \{ d_{\mathfrak{M}}(x, y) \mid 1 \leq i \leq \nu, x, y \in \overline{U}_i, d_{\mathfrak{X}}(\pi_i(x), \pi_i(y)) \geq \epsilon \} \}$. \square

LEMMA 2.9. [20] *There exists a continuous increasing function ρ_{τ} (the modulus of continuity for the sections τ_i) such that:*

$$(4) \quad \forall 1 \leq i \leq \nu \text{ and } w, w' \in \mathfrak{T}_i \quad , \quad d_{\mathfrak{X}}(w, w') < \rho_{\tau}(\epsilon) \implies d_{\mathfrak{M}}(\tau_i(w), \tau_i(w')) < \epsilon \text{ .}$$

Proof. Set $\rho_{\tau}(\epsilon) = \min \{ \epsilon, \min \{ d_{\mathfrak{X}}(w, w') \mid 1 \leq i \leq \nu, w, w' \in \mathfrak{T}_i, d_{\mathfrak{M}}(\tau_i(w), \tau_i(w')) \geq \epsilon \} \}$. \square

Finally, we introduce two additional constants, derived from the Lebesgue number $\epsilon_{\mathcal{U}}$ chosen above.

The first is derived from a “converse” to the modulus function ρ_{π} . Set:

$$(5) \quad \epsilon_{\mathcal{U}}^{\mathcal{T}} = \max \{ \epsilon \mid \forall 1 \leq i \leq \nu, \forall x \in \overline{U}_i, D_{\mathfrak{M}}(x, \epsilon_{\mathcal{U}}/2) \subset \overline{U}_i, D_{\mathfrak{X}}(\pi_i(x), \epsilon) \subset \pi_i(D_{\mathfrak{M}}(x, \epsilon_{\mathcal{U}}/2)) \} \text{ .}$$

Note that $\epsilon_{\mathcal{U}}^{\mathcal{T}} \geq \rho_{\tau}(\epsilon_{\mathcal{U}}/2)$.

For $y \in \mathfrak{M}$ recall that $D_{\mathcal{F}}(y, \epsilon)$ is the closed ball of radius ϵ for the leafwise metric. Introduce a form of “leafwise Lebesgue number”, defined by

$$(6) \quad \epsilon_{\mathcal{U}}^{\mathcal{F}} = \min \{ \epsilon_{\mathcal{U}}^{\mathcal{F}}(y) \mid \forall y \in \mathfrak{M} \} \text{ , } \epsilon_{\mathcal{U}}^{\mathcal{F}}(y) = \max \{ \epsilon \mid D_{\mathcal{F}}(y, \epsilon) \subset D_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/4) \} \text{ .}$$

Thus, for all $y \in \mathfrak{M}$, $D_{\mathcal{F}}(y, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset D_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/4)$. Note that for all $r > 0$ and $z' \in D_{\mathcal{F}}(z, \epsilon_{\mathcal{U}}^{\mathcal{F}})$, the triangle inequality implies that $D_{\mathfrak{M}}(z', r) \subset D_{\mathfrak{M}}(z, r + \epsilon_{\mathcal{U}}/4)$.

3. HOLONOMY OF FOLIATED SPACES

The holonomy pseudogroup of a smooth foliated manifold (M, \mathcal{F}) generalizes the discrete cascade associated to a section of a flow. The holonomy pseudogroup for a matchbox manifold $(\mathfrak{M}, \mathcal{F})$ is defined analogously, although there are delicate issues of domains which must be considered.

A pair of indices (i, j) , $1 \leq i, j \leq \nu$, is said to be *admissible* if the *open* coordinate charts satisfy $U_i \cap U_j \neq \emptyset$. For (i, j) admissible, define $\mathfrak{D}_{i,j} = \pi_i(U_i \cap U_j) \subset \mathfrak{T}_i \subset \mathfrak{X}$. Then the closure $\overline{\mathfrak{D}_{i,j}} = \pi_i(\overline{U_i \cap U_j})$. The regularity of foliation charts imply that plaques are either disjoint, or have connected intersection. This implies that there is a well-defined homeomorphism $h_{j,i}: \mathfrak{D}_{i,j} \rightarrow \mathfrak{D}_{j,i}$

with domain $D(h_{j,i}) = \mathfrak{D}_{i,j}$ and range $R(h_{j,i}) = \mathfrak{D}_{j,i}$. The map $h_{j,i}$ admits a continuous extension to $\overline{h_{j,i}}: \overline{\mathfrak{D}_{i,j}} \rightarrow \overline{\mathfrak{D}_{j,i}}$.

The maps $\mathcal{G}_{\mathcal{F}}^{(1)} = \{h_{j,i} \mid (i,j) \text{ admissible}\}$ are the transverse change of coordinates defined by the foliation charts. By definition they satisfy $h_{i,i} = Id$, $h_{i,j}^{-1} = h_{j,i}$, and if $U_i \cap U_j \cap U_k \neq \emptyset$ then $h_{k,j} \circ h_{j,i} = h_{k,i}$ on their common domain of definition. The *holonomy pseudogroup* $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is the topological pseudogroup modeled on \mathfrak{X} generated by compositions of the elements of $\mathcal{G}_{\mathcal{F}}^{(1)}$.

A sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ is *admissible*, if each pair $(i_{\ell-1}, i_\ell)$ is admissible for $1 \leq \ell \leq \alpha$, and the composition

$$(7) \quad h_{\mathcal{I}} = h_{i_\alpha, i_{\alpha-1}} \circ \dots \circ h_{i_1, i_0}$$

has non-empty domain. The domain $D(h_{\mathcal{I}})$ is the *maximal open subset* of $\mathfrak{D}_{i_0, i_1} \subset \mathfrak{T}_{i_0}$ for which the compositions are defined.

Given any open subset $U \subset D(h_{\mathcal{I}})$ we obtain a new element $h_{\mathcal{I}}|U \in \mathcal{G}_{\mathcal{F}}$ by restriction. Introduce

$$(8) \quad \mathcal{G}_{\mathcal{F}}^* = \{h_{\mathcal{I}}|U \mid \mathcal{I} \text{ admissible \& } U \subset D(h_{\mathcal{I}})\} \subset \mathcal{G}_{\mathcal{F}}.$$

The range of $g = h_{\mathcal{I}}|U$ is the open set $R(g) = h_{\mathcal{I}}(U) \subset \mathfrak{T}_{i_\alpha} \subset \mathfrak{X}$. Note that each map $g \in \mathcal{G}_{\mathcal{F}}^*$ admits a continuous extension $\bar{g}: \overline{D(g)} = \overline{U} \rightarrow \mathfrak{T}_{i_\alpha}$.

We introduce the standard notation for the orbits of the pseudogroup $\mathcal{G}_{\mathcal{F}}$, where for $w \in \mathfrak{X}$, set

$$(9) \quad \mathcal{O}(w) = \{g(w) \mid g \in \mathcal{G}_{\mathcal{F}}^*, w \in D(g)\} \subset \mathfrak{T}_*.$$

Given an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$, for each $0 \leq \ell \leq \alpha$, set $\mathcal{I}_\ell = (i_0, i_1, \dots, i_\ell)$ and

$$(10) \quad h_{\mathcal{I}_\ell} = h_{i_\ell, i_{\ell-1}} \circ \dots \circ h_{i_1, i_0}.$$

Given $\xi \in D(h_{\mathcal{I}})$ we adopt the notation $\xi_\ell = h_{\mathcal{I}_\ell}(\xi) \in \mathfrak{T}_{i_\ell}$. So $\xi_0 = \xi$ and $h_{\mathcal{I}}(\xi) = \xi_\alpha$.

Given $\xi \in D(h_{\mathcal{I}})$, let $x = x_0 = \tau_{i_0}(\xi_0) \in L_x$. Introduce the *plaque chain*

$$\mathcal{P}_{\mathcal{I}}(\xi) = \{\mathcal{P}_{i_0}(\xi_0), \mathcal{P}_{i_1}(\xi_1), \dots, \mathcal{P}_{i_\alpha}(\xi_\alpha)\}.$$

For each $0 \leq \ell < \alpha$, we have $\text{int}(\mathcal{P}_{i_\ell}(\xi_\ell)) \cap \text{int}(\mathcal{P}_{i_{\ell+1}}(\xi_{\ell+1})) \neq \emptyset$. Moreover, each $\mathcal{P}_{i_\ell}(\xi_\ell)$ is a strongly convex subset of the leaf L_x in the leafwise metric $d_{\mathcal{F}}$. Recall that $\mathcal{P}_{i_\ell}(x_\ell) = \mathcal{P}_{i_\ell}(\xi_\ell)$, so we also adopt the notation $\mathcal{P}_{\mathcal{I}}(x) \equiv \mathcal{P}_{\mathcal{I}}(\xi)$.

Intuitively, a plaque chain $\mathcal{P}_{\mathcal{I}}(\xi)$ is a sequence of successively overlapping convex “tiles” in L_0 starting at $x_0 = \tau_{i_0}(\xi_0)$, ending at $x_\alpha = \tau_{i_\alpha}(\xi_\alpha)$, and with each $\mathcal{P}_{i_\ell}(\xi_\ell)$ “centered” on the point $x_\ell = \tau_{i_\ell}(\xi_\ell)$.

3.1. Leafwise path holonomy. A standard construction in foliation theory, introduced by Poincaré for sections to flows, and developed for foliations by Reeb [52] (see also [37], [16], [17, Chapter 2]) associates to a leafwise path γ a holonomy map h_γ . We describe this construction below, paying particular attention to domains and metric estimates, which will play a crucial role later in the proof of main theorems.

Let \mathcal{I} be an admissible sequence. For $w \in D(h_{\mathcal{I}})$, we say that (\mathcal{I}, w) *covers* γ , if the domain of γ admits a partition $0 = s_0 < s_1 < \dots < s_\alpha = 1$ such that the plaque chain $\mathcal{P}_{\mathcal{I}}(w) = \{\mathcal{P}_{i_0}(w_0), \mathcal{P}_{i_1}(w_1), \dots, \mathcal{P}_{i_\alpha}(w_\alpha)\}$ satisfies

$$(11) \quad \gamma([s_\ell, s_{\ell+1}]) \subset \text{int}(\mathcal{P}_{i_\ell}(w_\ell)), \quad 0 \leq \ell < \alpha, \quad \& \quad \gamma(1) \in \text{int}(\mathcal{P}_{i_\alpha}(w_\alpha)).$$

It follows that $w_0 = \pi_{i_0}(\gamma(0)) \in D(h_{\mathcal{I}})$.

Now suppose we have two admissible sequences, $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ and $\mathcal{J} = (j_0, j_1, \dots, j_\beta)$, such that both (\mathcal{I}, w) and (\mathcal{J}, v) cover the leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$. Then

$$\gamma(0) \in \text{int}(\mathcal{P}_{i_0}(w_0)) \cap \text{int}(\mathcal{P}_{j_0}(v_0)) \quad , \quad \gamma(1) \in \text{int}(\mathcal{P}_{i_\alpha}(w_\alpha)) \cap \text{int}(\mathcal{P}_{j_\beta}(v_\beta))$$

Thus both (i_0, j_0) and (i_α, j_β) are admissible, and $v_0 = h_{j_0, i_0}(w_0)$, $w_\alpha = h_{i_\alpha, j_\beta}(v_\beta)$.

PROPOSITION 3.1. [20] *The maps $h_{\mathcal{I}}$ and $h_{i_\alpha, j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0, i_0}$ agree on their common domains.*

3.2. Admissible sequences. Given a leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$, we next construct an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ with $w \in D(h_{\mathcal{I}})$ so that (\mathcal{I}, w) covers γ , and has “uniform domains”.

Inductively, choose a partition of the interval $[0, 1]$, $0 = s_0 < s_1 < \dots < s_\alpha = 1$ such that for each $0 \leq \ell \leq \alpha$, $\gamma([s_\ell, s_{\ell+1}]) \subset D_{\mathcal{F}}(x_\ell, \epsilon_{\mathcal{U}}^{\mathcal{F}})$ where $x_\ell = \gamma(s_\ell)$. As a notational convenience, we have let $s_{\alpha+1} = s_\alpha$, so that $\gamma([s_\alpha, s_{\alpha+1}]) = x_\alpha$. Note that we can choose $s_{\ell+1}$ to be the largest value such that $d_{\mathcal{F}}(\gamma(s_\ell), \gamma(t)) \leq \epsilon_{\mathcal{U}}^{\mathcal{F}}$ for all $s_\ell \leq t \leq s_{\ell+1}$. Thus, we can assume $\alpha \leq 1 + \|\gamma\|/\epsilon_{\mathcal{U}}^{\mathcal{F}}$.

For each $0 \leq \ell \leq \alpha$, choose an index $1 \leq i_\ell \leq \nu$ so that $B_{\mathfrak{M}}(x_\ell, \epsilon_{\mathcal{U}}) \subset U_{i_\ell}$. Note that, for all $s_\ell \leq t \leq s_{\ell+1}$, $B_{\mathfrak{M}}(\gamma(t), \epsilon_{\mathcal{U}}/2) \subset U_{i_\ell}$, so that $x_{\ell+1} \in U_{i_\ell} \cap U_{i_{\ell+1}}$. It follows that $\mathcal{I}_\gamma = (i_0, i_1, \dots, i_\alpha)$ is an admissible sequence. Set $h_\gamma = h_{\mathcal{I}_\gamma}$. Then $h_\gamma(w) = w'$, where $w = \pi_{i_0}(x_0)$ and $w' = \pi_{i_\alpha}(x_\alpha)$.

The construction of the admissible sequence \mathcal{I}_γ above has the important property that $h_{\mathcal{I}_\gamma}$ is the composition of generators of $\mathcal{G}_{\mathcal{F}}^*$ which have a uniform lower bound estimate $\epsilon_{\mathcal{U}}^{\mathcal{T}}$ on the radii of the metric balls centered at the orbit, which are contained in their domains, with $\epsilon_{\mathcal{U}}^{\mathcal{T}}$ independent of γ . To see this, let $0 \leq \ell < \alpha$, and note that $x_{\ell+1} \in D_{\mathcal{F}}(x_{\ell+1}, \epsilon_{\mathcal{U}}^{\mathcal{F}})$ implies that for some $s_\ell < s'_{\ell+1} < s_{\ell+1}$, we have that $\gamma([s'_{\ell+1}, s_{\ell+1}]) \subset D_{\mathcal{F}}(x_{\ell+1}, \epsilon_{\mathcal{U}}^{\mathcal{F}})$. Hence,

$$(12) \quad B_{\mathfrak{M}}(\gamma(t), \epsilon_{\mathcal{U}}/2) \subset U_{i_\ell} \cap U_{i_{\ell+1}}, \text{ for all } s'_{\ell+1} \leq t \leq s_{\ell+1}.$$

Then for all $s'_{\ell+1} \leq t \leq s_{\ell+1}$, the uniform estimate defining $\epsilon_{\mathcal{U}}^{\mathcal{T}} > 0$ in (5) implies that

$$(13) \quad B_{\mathfrak{X}}(\pi_{i_\ell}(\gamma(t)), \epsilon_{\mathcal{U}}^{\mathcal{T}}) \subset \mathfrak{D}_{i_\ell, i_{\ell+1}} \quad \& \quad B_{\mathfrak{X}}(\pi_{i_{\ell+1}}(\gamma(t)), \epsilon_{\mathcal{U}}^{\mathcal{T}}) \subset \mathfrak{D}_{i_{\ell+1}, i_\ell}.$$

For the admissible sequence $\mathcal{I}_\gamma = (i_0, i_1, \dots, i_\alpha)$, recall that $x_\ell = \gamma(s_\ell)$ and we set $w_\ell = \pi_{i_\ell}(x_\ell)$. Then by the definition (7) of $h_{\mathcal{I}_\gamma}$ the condition (13) implies that $D_{\mathfrak{X}}(w_\ell, \epsilon_{\mathcal{U}}^{\mathcal{T}}) \subset D(h_\ell)$.

There is a converse to the above construction, which associates to an admissible sequence a leafwise path. Let $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ be admissible, with corresponding holonomy map $h_{\mathcal{I}}$, and choose $w \in D(h_{\mathcal{I}})$ with $x = \tau_{i_0}(w)$.

For each $1 \leq \ell \leq \alpha$, recall that $\mathcal{I}_\ell = (i_0, i_1, \dots, i_\ell)$, and let $h_{\mathcal{I}_\ell}$ denote the corresponding holonomy map. For $\ell = 0$, let $\mathcal{I}_0 = (i_0, i_0)$. Note that $h_{\mathcal{I}_\alpha} = h_{\mathcal{I}}$ and $h_{\mathcal{I}_0} = \text{Id}: \mathfrak{T}_0 \rightarrow \mathfrak{T}_0$.

For each $0 \leq \ell \leq \alpha$, set $w_\ell = h_{\mathcal{I}_\ell}(w)$ and $x_\ell = \tau_{i_\ell}(w_\ell)$. By assumption, for $\ell > 0$, there exists $z_\ell \in \mathcal{P}_{\ell-1}(w_{\ell-1}) \cap \mathcal{P}_\ell(w_\ell)$.

Let $\gamma_\ell: [(\ell-1)/\alpha, \ell/\alpha] \rightarrow L_{x_0}$ be the leafwise piecewise geodesic segment from $x_{\ell-1}$ to z_ℓ to x_ℓ . Define the leafwise path $\gamma_{\mathcal{I}}^x: [0, 1] \rightarrow L_{x_0}$ from x_0 to x_α to be the concatenation of these paths. If we then cover $\gamma_{\mathcal{I}}^x$ by the charts determined by the given admissible sequence \mathcal{I} , it follows that $h_{\mathcal{I}} = h_{\gamma_{\mathcal{I}}^x}$.

Thus, given an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ and $w \in D(h_{\mathcal{I}})$ with $w' = h_{\mathcal{I}}(w)$, the choices above determine an initial chart φ_{i_0} with “starting point” $x = \tau_{i_0}(w) \in U_{i_0} \subset \mathfrak{M}$. Similarly, there is a terminal chart φ_{i_α} with “terminal point” $x' = \tau_{i_\alpha}(w') \in U_{i_\alpha} \subset \mathfrak{M}$. The leafwise path $\gamma_{\mathcal{I}}^x$ constructed above starts at x , ends at x' , and has image contained in the plaque chain $\mathcal{P}_{\mathcal{I}}(x)$.

On the other hand, if we start with a leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$, then the initial point $x = \gamma(a)$ and the terminal point $x' = \gamma(b)$ are both well-defined. However, there need not be a unique index j_0 such that $x \in U_{j_0}$ and similarly for the index j_β such that $x' \in U_{j_\beta}$. Thus, when one constructs an admissible sequence $\mathcal{J} = (j_0, \dots, j_\beta)$ from γ , the initial and terminal charts need not be well-defined. That is, in fact, the essence of Proposition 3.1, which proved that

$$h_{\mathcal{I}}|U = h_{i_\alpha, j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0, i_0}|U \quad \text{for} \quad U = D(h_{\mathcal{I}}) \cap D(h_{i_\alpha, j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0, i_0}).$$

We conclude this discussion with a trivial observation, and an application which yields a key technical point, that the holonomy along a path is independent of “small deformations” of the path.

The observation is this. Let $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ be admissible, with associated holonomy map $h_{\mathcal{I}}$. Given $w, u \in D(h_{\mathcal{I}})$, then the germs of $h_{\mathcal{I}}$ at w and u admit a common extension, namely $h_{\mathcal{I}}$. Thus, if γ, γ' are leafwise paths defined as above from the plaque chains associated to (\mathcal{I}, w) and (\mathcal{I}, u) then the germinal holonomy maps along γ and γ' admit a common extension by Proposition 3.1. This is the basic idea behind the following technically useful result.

LEMMA 3.2. [20] *Let $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ be leafwise paths. Suppose that $x = \gamma(0), x' = \gamma'(0) \in U_i$ and $y = \gamma(1), y' = \gamma'(1) \in U_j$. If $d_{\mathfrak{M}}(\gamma(t), \gamma'(t)) \leq \epsilon_U/4$ for all $0 \leq t \leq 1$, then the induced holonomy maps $h_\gamma, h_{\gamma'}$ agree on their common domain $D(h_\gamma) \cap D(h_{\gamma'}) \subset \mathfrak{T}_i$.*

In particular, if curves γ, γ' are sufficiently close, then they define holonomy maps which have a common extension.

3.3. Homotopy independence. Two leafwise paths $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ are homotopic if there exists a family of leafwise paths $\gamma_s: [0, 1] \rightarrow \mathfrak{M}$ with $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$. We are most interested in the special case when $\gamma(0) = \gamma'(0) = x$ and $\gamma(1) = \gamma'(1) = y$. Then γ and γ' are *endpoint-homotopic* if they are homotopic with $\gamma_s(0) = x$ for all $0 \leq s \leq 1$, and similarly $\gamma_s(1) = y$ for all $0 \leq s \leq 1$. Thus, the family of curves $\{\gamma_s(t) \mid 0 \leq s \leq 1\}$ are all contained in a common leaf L_x . The following property then follows from an inductive application of Lemma 3.2:

LEMMA 3.3. [20] *Let $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ be endpoint-homotopic leafwise paths. Then their holonomy maps h_γ and $h_{\gamma'}$ agree on some open subset $U \subset D(h_\gamma) \cap D(h_{\gamma'}) \subset \mathfrak{T}_*$. In particular, they determine the same germinal holonomy maps.*

The following is another consequence of the strongly convex property of the plaques:

LEMMA 3.4. [20] *Suppose that $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ are leafwise paths for which $\gamma(0) = \gamma'(0) = x$ and $\gamma(1) = \gamma'(1) = x'$, and suppose that $d_{\mathfrak{M}}(\gamma(t), \gamma'(t)) < \epsilon_U/2$ for all $a \leq t \leq b$. Then $\gamma, \gamma': [0, 1] \rightarrow \mathfrak{M}$ are endpoint-homotopic.*

Given $g \in \mathcal{G}_{\mathcal{F}}^*$ and $w \in D(g)$, let $[g]_w$ denote the germ of the map g at $w \in \mathfrak{T}_*$. Set

$$(14) \quad \Gamma_{\mathcal{F}}^w = \{[g]_w \mid g \in \mathcal{G}_{\mathcal{F}}^*, w \in D(g), g(w) = w\}.$$

Given $x \in U_i$ with $w = \pi_i(x) \in \mathfrak{T}_*$, the elements of $\Gamma_{\mathcal{F}}^w$ form a group, and by Lemma 3.3 there is a well-defined homomorphism $h_{\mathcal{F}, x}: \pi_1(L_x, x) \rightarrow \Gamma_{\mathcal{F}}^w$ which is called the *holonomy group* of \mathcal{F} at x .

3.4. Non-trivial holonomy. Note that if $y \in L_x$ then the homomorphism $h_{\mathcal{F}, y}$ is conjugate (by an element of $\mathcal{G}_{\mathcal{F}}^*$) to the homomorphism $h_{\mathcal{F}, x}$. A leaf L is said to have *non-trivial germinal holonomy* if for some $x \in L$, the homomorphism $h_{\mathcal{F}, x}$ is non-trivial. If the homomorphism $h_{\mathcal{F}, x}$ is trivial, then we say that L_x is a *leaf without holonomy*. This property depends only on L , and not the basepoint $x \in L$. The foliated space \mathfrak{M} is said to be *without holonomy* if for every $x \in M$, the leaf L_x is without germinal holonomy.

LEMMA 3.5. [20] *Let \mathfrak{M} be a foliated space, and L_x a leaf without holonomy. Fix a regular covering for \mathfrak{M} as above, and let $w \in \mathfrak{T}_*$ be the local projection of a point in L_x . Given plaques chains \mathcal{I}, \mathcal{J} such that $w \in \text{Dom}(h_{\mathcal{I}}) \cap \text{Dom}(h_{\mathcal{J}})$ with $h_{\mathcal{I}}(w) = w' = h_{\mathcal{J}}(w)$, then $h_{\mathcal{I}}$ and $h_{\mathcal{J}}$ have the same germinal holonomy at w . Thus, for each $w' \in \mathcal{O}(w)$ in the $\mathcal{G}_{\mathcal{F}}^*$ orbit of w , there is a well-defined holonomy germ $h_{w, w'}$.*

Proof. The composition $g = h_{\mathcal{J}}^{-1} \circ h_{\mathcal{I}}$ satisfies $g(w) = w$, so by assumption there is some open neighborhood $w \in U$ for which $g|_U$ is the trivial map. That is, $h_{\mathcal{I}}|_U = h_{\mathcal{J}}|_U$. \square

Finally, we recall a basic result of Epstein, Millett and Tischler [28] for foliated manifolds, whose proof applies verbatim in the case of foliated spaces.

THEOREM 3.6. *The union of all leaves without holonomy in a foliated space \mathfrak{M} is a dense G_δ subset of \mathfrak{M} . In particular, there exists at least one leaf without germinal holonomy.*

4. DYNAMICS OF MATCHBOX MANIFOLDS

We first recall several important classical definitions from topological dynamics, adapted to the case of matchbox manifolds, and several results concerning their dynamical properties from [20].

DEFINITION 4.1. *The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is equicontinuous if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $g \in \mathcal{G}_{\mathcal{F}}^*$, if $w, w' \in D(g)$ and $d_{\mathfrak{X}}(w, w') < \delta$, then $d_{\mathfrak{X}}(g(w), g(w')) < \epsilon$.*

DEFINITION 4.2. *The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is distal if for all $w, w' \in \mathfrak{T}_*$, if $w \neq w'$ then there exists $\delta_{w, w'} > 0$ such that for all $g \in \mathcal{G}_{\mathcal{F}}^*$ with $w, w' \in D(g)$, then $d_{\mathfrak{X}}(g(w), g(w')) \geq \delta_{w, w'}$.*

Distal and equicontinuous pseudogroups are closely related [5, 8, 27, 31, 42], while the following notion is their direct opposite.

DEFINITION 4.3. *The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is expansive, or more properly ϵ -expansive, if there exists $\epsilon > 0$ such that for all $w, w' \in \mathfrak{T}_*$, there exists $g \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in D(g)$ such that $d_{\mathfrak{X}}(g(w), g(w')) \geq \epsilon$.*

Equicontinuity for $\mathcal{G}_{\mathcal{F}}$ gives *uniform* control over the domains of arbitrary compositions of generators.

PROPOSITION 4.4. [20] *Assume the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is equicontinuous. Then there exists $\delta_{\mathcal{U}}^T > 0$ such that for every leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$, there is a corresponding admissible sequence $\mathcal{I}_{\gamma} = (i_0, i_1, \dots, i_{\alpha})$ so that $B_{\mathfrak{X}}(w_0, \delta_{\mathcal{U}}^T) \subset D(h_{\mathcal{I}_{\gamma}})$, where $x = \gamma(0)$ and $w_0 = \pi_{i_0}(x)$.*

Moreover, for all $0 < \epsilon_1 \leq \epsilon_{\mathcal{U}}^T$ there exists $0 < \delta_1 \leq \delta_{\mathcal{U}}^T$ independent of the path γ , such that $h_{\mathcal{I}_{\gamma}}(D_{\mathfrak{X}}(w_0, \delta_1)) \subset D_{\mathfrak{X}}(w', \epsilon_1)$ where $w' = \pi_{i_{\alpha}}(\gamma(1))$.

Thus, $\mathcal{G}_{\mathcal{F}}^$ is equicontinuous as a family of local group actions.*

We next recall several results concerning minimal matchbox manifolds. First recall:

DEFINITION 4.5. *A foliated space \mathfrak{M} is minimal if each leaf $L \subset \mathfrak{M}$ is dense.*

The following is an immediate consequence of the definitions:

LEMMA 4.6. *A foliated space \mathfrak{M} is minimal if and only if for some regular covering of \mathfrak{M} , the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of \mathcal{F} is minimal; that is, for all $w \in \mathfrak{T}_*$, the $\mathcal{G}_{\mathcal{F}}$ orbit $\mathcal{O}(w)$ of w is dense.*

The following result is, at first glance, very surprising. It has been previously shown for flows [1] and \mathbb{R}^n -actions [18]. The proof of it is given in detail in [20, Section 4.1], and fundamentally uses the conclusions of Proposition 4.4.

THEOREM 4.7. [20] *If \mathfrak{M} is an equicontinuous matchbox manifold, then \mathfrak{M} is minimal.*

One of the main technical results of the work [20] is a much stronger version of Theorem 4.7, which says that not only is an equicontinuous matchbox manifold minimal, but it admits finite codings of its orbits with arbitrarily fine equivalence classes. The proof of the following theorem is given in [20, Section 6].

THEOREM 4.8. [20] *Let \mathfrak{M} is an equicontinuous matchbox manifold, and $w_0 \in \mathfrak{T}_*$ a basepoint. Then there exists a descending chain of clopen subsets*

$$\cdots \subset V_{\ell+1} \subset V_{\ell} \subset \cdots \subset V_0 \subset \mathfrak{T}_*$$

such that for all $\ell \geq 0$, $w_0 \in V_{\ell}$ and $\text{diam}_{\mathfrak{X}}(V_{\ell}) < \delta_{\mathcal{U}}^T/2^{\ell}$.

Moreover, each V_{ℓ} is $\mathcal{G}_{\mathcal{F}}$ -invariant in the following sense: if γ is a path with initial point $\gamma(0) \in V_{\ell}$, then the holonomy map h_{γ} satisfies $V_{\ell} \subset \text{Dom}(h_{\gamma})$, and if $h_{\gamma}(V_{\ell}) \cap V_{\ell} \neq \emptyset$, then $h_{\gamma}(V_{\ell}) = V_{\ell}$. \square

It follows that the collection $\{h_\gamma(V_\ell) \mid \gamma(0) \in V_\ell\}$ of subsets of the transverse space \mathfrak{T}_* forms a finite clopen partition, and these sets are permuted by the action of the holonomy pseudogroup.

If a foliation is expansive, then the domains of arbitrary compositions of generators for its holonomy typically do not admit uniform estimates as in Proposition 4.4. However, the compactness of \mathfrak{M} implies there is a uniform estimate on the size of the domain of a holonomy map formed from a bounded number of compositions used to define it, which is used in the proof of the Theorem 1.3.

Recall that the path length in the $d_{\mathcal{F}}$ metric of the piecewise C^1 -curve $\gamma(t)$ is denoted by $\|\gamma\|$.

PROPOSITION 4.9. *For each $\epsilon > 0$ and $r > 0$, there exists $0 < \delta(\epsilon, r) \leq \epsilon$ so that for any piecewise smooth leafwise path $\gamma: [0, 1] \rightarrow \mathfrak{M}$ with $\|\gamma\| \leq r$, then there exists an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ such that (\mathcal{I}, w) covers γ with:*

- (1) $w_0 = \pi_{i_0}(\gamma(0)) \in D(h_{\mathcal{I}})$ and $D_{\mathfrak{X}}(w_0, \delta(\epsilon, r)) \subset D(h_{\mathcal{I}})$;
- (2) $h_{\mathcal{I}}(D_{\mathfrak{X}}(w_0, \delta(\epsilon, r))) \subset D_{\mathfrak{X}}(w', \epsilon)$ where $w' = \pi_{i_\alpha}(\gamma(1))$.

Proof. By the arguments of Section 3.2, there exists an admissible sequence $\mathcal{I} = (i_0, i_1, \dots, i_\alpha)$ with $w \in D(h_{\mathcal{I}})$ and $\alpha \leq 1 + \|\gamma\|/\epsilon_{\mathcal{U}}^{\mathcal{F}} \leq 1 + r/\epsilon_{\mathcal{U}}^{\mathcal{F}}$ where $\epsilon_{\mathcal{U}}^{\mathcal{F}} > 0$ is defined by (6).

We estimate the size of the domain $D(h_{\mathcal{I}})$. For each $0 \leq \ell \leq \alpha$, we have $\gamma([s_\ell, s_{\ell+1}]) \subset D_{\mathcal{F}}(x_\ell, \epsilon_{\mathcal{U}}^{\mathcal{F}})$ where $x_\ell = \gamma(s_\ell)$. Moreover, for the associated admissible sequence $\mathcal{I}_\gamma = (i_0, i_1, \dots, i_\alpha)$, we have that for all $0 \leq t \leq 1$, $B_{\mathfrak{M}}(\gamma(t), \frac{1}{2}\epsilon_{\mathcal{U}}) \subset U_{i_\ell}$.

The proof will be by downward induction. Let $w_\ell = \pi_{i_\ell}(x_\ell)$ and set $\mathcal{I}_\ell = (i_0, i_1, \dots, i_\ell)$ with corresponding holonomy map $h_{\mathcal{I}_\ell}$. Then $h_{\mathcal{I}_\ell}(w_0) = w_\ell$. Let $h_\ell = h_{i_{\ell+1}, i_\ell}$ so that $h_\ell \circ h_{\mathcal{I}_\ell} = h_{\mathcal{I}_{\ell+1}}$.

For every admissible pair (i, j) the holonomy homeomorphism $h_{j,i}$ is uniformly continuous as it has compact domain. Since there is only a finite number of distinct non-empty intersections $U_i \cap U_j$, for every $\epsilon > 0$ there exists a $0 < \delta_\epsilon \leq \epsilon$ such that for every admissible pair (i, j) if $w, w' \in D(h_{j,i})$ and $d_{\mathfrak{X}}(w, w') < \delta_\epsilon$ then $d_{\mathfrak{X}}(h_{j,i}(w), h_{j,i}(w')) < \epsilon$.

Recall $\epsilon_{\mathcal{U}}^{\mathcal{F}} > 0$ as defined by (5). Given $\epsilon > 0$, set $\epsilon_\alpha = \min\{\epsilon_{\mathcal{U}}^{\mathcal{F}}/2, \epsilon\}$. Now proceed by downward induction. For $0 < \ell \leq \alpha$ assume that ϵ_ℓ has been defined. Then denote $\delta_\ell = \delta_{\epsilon_\ell}$ and $\epsilon_{\ell-1} = \delta_\ell$ as defined using equicontinuity as above.

By the choice of the covering of the admissible sequence \mathcal{I} , we have $D_{\mathfrak{X}}(w_\ell, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset D(h_\ell)$ and so $D_{\mathfrak{X}}(w_\ell, \delta_\ell) \subset D(h_\ell)$, and by the choice of δ_ℓ we have $D_{\mathfrak{X}}(w_\ell, \delta_\ell) \subset D(h_\ell \circ h_{\ell+1} \circ \dots \circ h_\alpha)$. Then $\delta(\epsilon, r) = \delta_1$ satisfies the required conditions. \square

5. FOLIATED MICROBUNDLES AND REEB STRUCTURE THEOREM

The Reeb Stability Theorem [16, 17, 52, 60] is one of the fundamental results of foliation theory. It states that for a compact leaf $L \subset M$ in a foliated manifold M with finite germinal holonomy group, there exists an open neighborhood $L \subset U$ which is a union of leaves of \mathcal{F} , and each leaf of $\mathcal{F}|_U$ is a finite covering of L . In particular, for a foliation defined by a flow, if the germinal holonomy of a periodic orbit is finite, then nearby orbits are also periodic and have bounded length. The most general version of these ideas is formulated in terms of the “foliated microbundle” associated to the holonomy covering of a leaf in a foliated space. (See Milnor [46] for a discussion of the concept of foliated microbundles for manifolds). This general formulation admits a generalization to matchbox manifolds, which we give in this section.

5.1. Nets and covers. Recall that we assume there is a fixed regular covering \mathcal{U} for \mathfrak{M} , as in Proposition 2.6 which we can assume to be finite. Let $w_0 \in \text{int}(\mathfrak{T}_1)$ be a fixed base-point. Let $x_0 = \tau_1(w_0) \in U_1$ and L_0 be the leaf through x_0 . Let $h_{\mathcal{F}, x_0}: \pi_1(L_0, x_0) \rightarrow \Gamma_{\mathcal{F}}^{w_0}$ denote the holonomy representation, where $\Gamma_{\mathcal{F}}^{w_0}$ is defined by (14). Since the map $h_{\mathcal{F}, x_0}$ is a homomorphism, its kernel $\mathfrak{K}_0 \subset \pi_1(L_{x_0}, x_0)$ is a normal subgroup, and the covering $\Pi: \tilde{L}_0 \rightarrow L_0$ associated to \mathfrak{K}_0 is regular.

Choose $\tilde{x}_0 \in \tilde{L}_0$ such that $\pi(\tilde{x}_0) = x_0$. By definition, given any closed path $\tilde{\gamma}: [0, 1] \rightarrow \tilde{L}_0$ with basepoint $\tilde{x}_0 = \tilde{\gamma}(0) = \tilde{\gamma}(1)$, the image of $\tilde{\gamma}$ in L_0 has trivial germinal holonomy as a leafwise path in \mathfrak{M} . It follows that the holonomy map defined by a path $\tilde{\gamma}$ in \tilde{L}_0 starting at \tilde{x}_0 is determined by the endpoint $\tilde{\gamma}(1)$.

For any leaf $L \subset \mathfrak{M}$ and covering $\Pi: \tilde{L} \rightarrow L$, the leafwise Riemannian metric $d_{\mathcal{F}}$ on a leaf L lifts to a Riemannian metric $d_{\tilde{\mathcal{F}}}$ on \tilde{L} such that Π is a local isometry.

We next select a collection of points in \tilde{L}_0 which are sufficiently dense, so that homotopy classes of paths between the points capture all of the holonomy defined by the leaf L_0 , but not too close to each other, so that we have some freedom to work in a neighborhood of a point without affecting other points. Such a collection is described by the following definition.

DEFINITION 5.1. *Let (X, d_X) be a complete separable metric space. Given $0 < e_1 < e_2$, a subset $\mathcal{M} \subset X$ is a (e_1, e_2) -net (or Delaunay set) if:*

- (1) \mathcal{M} is e_1 -separated: for all $y \neq z \in \mathcal{M}$, $e_1 \leq d_X(y, z)$;
- (2) \mathcal{M} is e_2 -dense: for all $x \in X$, there exists some $z \in \mathcal{M}$ such that $d_X(x, z) \leq e_2$.

Given a leaf $L \subset \mathfrak{M}$, the intersection $L \cap \mathcal{T}$ is a countable set of points which, since \mathfrak{M} is compact, always satisfies the density condition (5.1.2) for some $e_2 > 0$, but need not satisfy the separation condition (5.1.1). Also, it is elementary that given a separable, complete metric space X and any $e_2 > 0$, there exists $0 < e_1 < e_2$ and a (e_1, e_2) -net $\mathcal{M} \subset X$. Next, given e_1 and e_2 chosen in accordance with constants defined in Section 2, we construct a (e_1, e_2) -net in the given leaf L_0 .

Recall that $\epsilon_{\mathcal{U}}^{\mathcal{F}}$ defined by (6) was chosen so that every leafwise disk of radius $\epsilon_{\mathcal{U}}^{\mathcal{F}}$ is contained in a metric ball of \mathfrak{M} of radius $\epsilon_{\mathcal{U}}/4$. That is, for all $y \in \mathfrak{M}$, $D_{\mathcal{F}}(y, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset D_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/4)$.

Let $e_2 = \epsilon_{\mathcal{U}}^{\mathcal{F}}/4$, then choose $\mathcal{M}_0 \subset L_0$ an (e_1, e_2) -net for L_0 for some $0 < e_1 < e_2$. We can assume without loss of generality that $x_0 \in \mathcal{M}_0$. Condition (5.1.2) implies that the collection of leafwise open disks $\{B_{\mathcal{F}}(z, \epsilon_{\mathcal{U}}^{\mathcal{F}}/2) \mid z \in \mathcal{M}_0\}$ is an open covering of L_0 .

We next construct a subcover of \mathcal{U} associated to \mathcal{M}_0 . For each $z \in \mathcal{M}_0$, choose an index $1 \leq i_z \leq \nu$ so that $B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$. Without loss of generality, we can assume that $B_{\mathfrak{M}}(x_0, \epsilon_{\mathcal{U}}) \subset U_1$. Then note that for all $z' \in D_{\mathcal{F}}(z, \epsilon_{\mathcal{U}}^{\mathcal{F}})$, we have $z' \in D_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}/4)$ so the triangle inequality implies that

$$(15) \quad D_{\mathcal{F}}(z', \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset D_{\mathfrak{M}}(z', \epsilon_{\mathcal{U}}/4) \subset D_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}/2) \subset B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}.$$

LEMMA 5.2. *If the leaf L_0 is dense, then the collection $\{U_{i_z} \mid z \in \mathcal{M}_0\}$ is a subcover for \mathfrak{M} , with Lebesgue number $\epsilon_{\mathcal{U}}/2$.*

Proof. First, we show that the collection is a covering of \mathfrak{M} . Let $y \in \mathfrak{M}$, then L_0 is dense so there exists $y' \in L_0$ with $d_{\mathfrak{M}}(y, y') < \epsilon_{\mathcal{U}}/4$. Let $z \in \mathcal{M}_0$ with $d_{\mathcal{F}}(y', z) \leq e_2 = \epsilon_{\mathcal{U}}^{\mathcal{F}}/4$. Then $y' \in D_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}/4)$ by (6), hence $y \in B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}/2) \subset U_{i_z}$ by (15).

Next, we show that $\epsilon_{\mathcal{U}}/2$ is a Lebesgue number for this covering. Let $y'' \in B_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/2)$ then the above implies that $y'' \in B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$ by the choice of i_z above. Thus, $B_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/2) \subset U_{i_z}$. \square

5.2. Foliated microbundle. We next give the construction of the foliated microbundle associated to the choices made above. Let $\tilde{\mathcal{M}}_0 = \Pi^{-1}(\mathcal{M}_0)$ which is a (e_1, e_2) -net for \tilde{L}_0 with the Riemannian metric lifted from L_0 . The points of $\tilde{\mathcal{M}}_0$ are denoted by \tilde{z} , and where \tilde{z} is a lift of $z \in \mathcal{M}_0$. In particular, $\tilde{x}_0 \in \tilde{\mathcal{M}}_0$ as $\Pi(\tilde{x}_0) = x_0 \in \mathcal{M}_0$.

For each $\tilde{z} \in \tilde{\mathcal{M}}_0$, set $\tilde{U}_{\tilde{z}} = \overline{U}_{i_z} \times \{\tilde{z}\}$. For $(x, \tilde{z}) \in \tilde{U}_{\tilde{z}}$ define $\Pi: \tilde{U}_{\tilde{z}} \rightarrow \overline{U}_{i_z}$ by $\Pi(x, \tilde{z}) = x$. For $\tilde{z} \neq \tilde{z}' \in \tilde{\mathcal{M}}_0$ with $\Pi(\tilde{z}) = \Pi(\tilde{z}') = z$, the sets $\tilde{U}_{\tilde{z}}$ and $\tilde{U}_{\tilde{z}'}$ are disjoint by definition, though their projections to \mathfrak{M} agree.

For $\tilde{z} \in \tilde{\mathcal{M}}_0$ and $\tilde{y} = (x, \tilde{z}) \in \tilde{U}_{\tilde{z}}$, let $\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{y}) = \mathcal{P}_{i_z}(x) \times \{\tilde{z}\}$ denote the plaque of $\tilde{U}_{\tilde{z}}$ containing \tilde{y} . If $x \in \mathcal{P}_{i_z}(z)$ then we identify $\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{y})$ with the plaque of \tilde{L}_0 containing \tilde{z} . Note that by choice, $D_{\tilde{L}_0}(\tilde{z}, \epsilon_{\mathcal{U}}^{\tilde{\mathcal{F}}}) \subset \tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z})$ for each $\tilde{z} \in \tilde{\mathcal{M}}_0$, so the collection $\{\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z}) \mid \tilde{z} \in \tilde{\mathcal{M}}_0\}$ is an open covering of \tilde{L}_0 .

One thinks of the plaques $\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z})$ as “convex tiles”, and the collection $\{\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z}) \mid \tilde{z} \in \tilde{\mathcal{M}}_0\}$ as a “tiling” of \tilde{L}_0 . The interiors of the plaques need not be disjoint, so this is not a proper tiling in the usual sense (for example see [7, 12], or [17, §11.3.C]).

DEFINITION 5.3. *The foliated microbundle over \tilde{L}_0 is the space*

$$(16) \quad \tilde{\mathfrak{N}}_0 = \bigcup_{\tilde{z} \in \tilde{\mathcal{M}}_0} \tilde{U}_{\tilde{z}} / \sim$$

where $\tilde{y} \in \tilde{U}_{\tilde{z}}$ and $\tilde{y}' \in \tilde{U}_{\tilde{z}'}$ are identified if $\Pi(\tilde{y}) = \Pi(\tilde{y}')$ and $\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z}) \cap \tilde{\mathcal{P}}_{\tilde{z}'}(\tilde{z}') \neq \emptyset$. The connected components of $\tilde{\mathfrak{N}}_0$ form the leaves of a foliation $\tilde{\mathcal{F}}$.

Informally, the space $\tilde{\mathfrak{N}}_0$ is simply the union of copies of all flow boxes associated as above to the points $\tilde{z} \in \tilde{\mathcal{M}}_0$ and identified in the obvious fashion to obtain a continuous covering of \tilde{L}_0 . The usefulness of the foliated microbundle as constructed, is that it provides a uniform setting for all of the holonomy maps for paths in the leaf \tilde{L}_0 .

Introduce the transversals to $\tilde{\mathcal{F}}$ which are the lifts of the transversals to \mathcal{F} . For each $\tilde{z} \in \tilde{\mathcal{M}}_0$, let $\mathfrak{T}_{\tilde{z}} = \mathfrak{T}_{i_z} \times \{\tilde{z}\}$. The composition $\tilde{\varphi}_{\tilde{z}} \equiv \varphi_{i_z} \circ \Pi: \tilde{U}_{\tilde{z}} \rightarrow [-1, 1]^n \times \mathfrak{T}_{\tilde{z}}$ defines a coordinate chart on $\tilde{\mathfrak{N}}_0$, making it into a foliated space with foliation denoted by $\tilde{\mathcal{F}}$. Let $\tilde{\pi}_{\tilde{z}}: \tilde{U}_{\tilde{z}} \rightarrow \mathfrak{T}_{\tilde{z}}$ be the normal coordinate, and $\tilde{\lambda}_{\tilde{z}}: \tilde{U}_{\tilde{z}} \rightarrow [-1, 1]^n$ be the leafwise coordinate.

Given $\tilde{z} \in \tilde{\mathcal{M}}_0$, subset $V \subset \mathfrak{T}_{\tilde{z}}$ and $\xi \in [-1, 1]^n$, we obtain a local section for $\tilde{\mathcal{F}}$ by

$$(17) \quad \tilde{\tau}_{\tilde{z}, \xi}: V \rightarrow \tilde{U}_{\tilde{z}}, \quad \tilde{\tau}_{\tilde{z}, \xi}(w) = \tilde{\varphi}_{\tilde{z}}^{-1}(\xi, w) = (\varphi_{i_z}^{-1}(\xi, w), \tilde{z}).$$

Note that while the core leaf \tilde{L}_0 of the foliated microbundle $\tilde{\mathfrak{N}}_0$ is a regular covering of the leaf $L_0 \subset \mathfrak{M}$, the projection of other leaves \tilde{L} of $\tilde{\mathcal{F}}$ may not be coverings, as the leaves of $\tilde{\mathcal{F}}$ may “escape” from the flow boxes defining $\tilde{\mathfrak{N}}_0$. Here is a simple example to illustrate this point.

EXAMPLE 5.4. Let \mathfrak{M} be a 2-dimensional matchbox manifold consisting of two leaves: a 1-ended non-compact cylinder L_1 (a cylinder with a “cap” on one side), and a toral leaf L_2 . Such an example can be obtained, for instance, by the construction of Kenyon and Ghys, as described in [33, 45]. Alternatively, one can construct \mathfrak{M} as the closed saturated subset of the Reeb foliation of a solid torus in \mathbb{R}^3 , given by the closure of a non-compact leaf. That is, \mathfrak{M} consists of the boundary toral leaf L_1 and a single leaf L_2 from the interior of the solid torus. Since \mathfrak{M} is compact, L_2 must accumulate on L_1 , i.e. $L_1 \subset \overline{L_2}$. Then L_1 has infinite holonomy with a generator represented by a loop γ , and the lift of γ to the holonomy cover \tilde{L}_1 is an embedded line. The leaf L_2 has no holonomy.

Let $\{U'_1, U'_2, \dots, U'_k\}$ be a regular open covering of L_1 , then each U'_i extends to an open foliated chart U_i for \mathfrak{M} , but their union cannot cover L_2 completely. It is necessary to add at least one additional foliated chart U_{k+1} , consisting of a single plaque, to cover the rest of L_2 given by the “cap” as earlier in the text of the example. Let \mathcal{M}_1 be a Delaunay set in L_1 . Let $\tilde{\mathfrak{N}}_0$ be the microbundle as in Definition 5.3. We notice that $\tilde{\mathfrak{N}}_0$ is non-compact and does not contain dense leaves. There is a leaf $\tilde{L}_1 \cong L_1$, which projects onto L_1 under Π , and a countable number of leaves homeomorphic to a 1-ended cylinder with boundary (without a cap on one side), one of which projects onto the toral leaf L_1 under Π , and every other one projects into L_1 . The dynamics of \mathcal{F} is expansive, as the plaques in the covering of L_1 are transversely forced apart along an infinite ray in L_2 . \square

5.3. Equicontinuous matchbox manifolds and Thomas tubes. We consider the properties of the foliated microbundles when \mathcal{F} has equicontinuous dynamics. The additional regularity imposed by equicontinuity implies that the leaves of $\tilde{\mathcal{F}}$ in $\tilde{\mathfrak{N}}_0$ are all coverings of leaves of \mathcal{F} in \mathfrak{M} .

A path $\tilde{\gamma}: [0, 1] \rightarrow \tilde{L}_0$ is said to be *nice*, if there exists a partition $a = s_0 < s_1 < \dots < s_\alpha = b$ such that for each $0 \leq \ell \leq \alpha$, the restriction $\tilde{\gamma}: [s_\ell, s_{\ell+1}] \rightarrow \tilde{L}_0$ is a geodesic segment between points $\tilde{z}_\ell = \tilde{\gamma}(s_\ell), \tilde{z}_{\ell+1} = \tilde{\gamma}(s_{\ell+1}) \in \tilde{\mathcal{M}}_0$ with $d_{\mathcal{F}}(\tilde{z}_\ell, \tilde{z}_{\ell+1}) < \epsilon_{\mathcal{U}}^{\mathcal{F}}$. Then $\tilde{\mathcal{I}} = (\tilde{z}_0, \dots, \tilde{z}_\alpha)$ is an admissible sequence for $\tilde{\mathcal{F}}$, and $\mathcal{I} = (i_{z_0}, \dots, i_{z_\alpha})$ is an admissible sequence for \mathcal{F} . The sequence $\tilde{\mathcal{I}}$ defines the holonomy maps $\tilde{h}_{\tilde{\mathcal{I}}}$ for $\tilde{\mathcal{F}}$, and \mathcal{I} defines the holonomy map $h_{\mathcal{I}}$ for \mathcal{F} . Clearly, $\tilde{h}_{\mathcal{I}}$ is just the lift of $h_{\mathcal{I}}$, and $h_{\mathcal{I}}$ is the holonomy map for the leafwise path $\gamma = \Pi \circ \tilde{\gamma}$ constructed in Section 3. As before, we note that $\tilde{h}_{\tilde{\mathcal{I}}}$ depends only on the endpoints of \mathcal{I} . For $\tilde{z} \in \tilde{\mathcal{M}}_0$ let $\tilde{h}_{\tilde{z}}$ denote the holonomy along some nice path $\tilde{\gamma}_{\tilde{z}}$ from \tilde{x}_0 to \tilde{z} , considered as a transformation of the space $\tilde{\mathfrak{T}}$, which is the disjoint union of the local transversals $\tilde{\mathfrak{T}}_{\tilde{z}}$. Let $h_{\tilde{z}}$ denote the holonomy along the path $\gamma_{\tilde{z}} = \Pi \circ \tilde{\gamma}_{\tilde{z}}$.

If \mathfrak{M} be an equicontinuous matchbox manifold, then by Theorem 4.8, for any $\epsilon > 0$, there exists a $\mathcal{G}_{\mathcal{F}}$ -invariant clopen subset $w_0 \in V \subset \mathfrak{T}_*$ satisfying $\text{diam}_{\mathfrak{X}}(V) < \epsilon$. For $h_\gamma \in \mathcal{G}_{\mathcal{F}}^*$ with $V \subset D(h_\gamma)$, set $V_\gamma = h_\gamma(V)$. For $\tilde{z} \in \tilde{\mathcal{M}}_0$ there is a nice path $\gamma_{\tilde{z}}$ from \tilde{z}_0 to \tilde{z} which defines a holonomy map denoted by $h_{\tilde{z}} \equiv h_{\gamma_{\tilde{z}}}$. Then for $\tilde{z} \in \tilde{\mathcal{M}}_0$ define

$$(18) \quad V_{\tilde{z}} = h_{\tilde{z}}(V) \subset \mathfrak{T}_{i_{\tilde{z}}}, \quad \tilde{V}_{\tilde{z}} = \tilde{h}_{\tilde{z}}(V) = V_{\tilde{z}} \times \{\tilde{z}\} \subset \tilde{\mathfrak{T}}_{\tilde{z}}.$$

The union of the sets $V_{\tilde{z}}$ is the saturation of V under the action of the pseudogroup $\mathcal{G}_{\mathcal{F}}^*$, and hence it forms a clopen partition of \mathfrak{T}_* . Introduce the local coordinate chart saturations of these sets:

$$(19) \quad \mathfrak{U}_{\tilde{z}}^V = \pi_{i_{\tilde{z}}}^{-1}(V_{\tilde{z}}) \subset \overline{U}_{i_{\tilde{z}}}, \quad \tilde{\mathfrak{U}}_{\tilde{z}}^V = \mathfrak{U}_{\tilde{z}}^V \times \{\tilde{z}\} \subset \tilde{U}_{\tilde{z}}.$$

Then $\mathfrak{U}_{\tilde{z}}^V$ is the union of the plaques in $\overline{U}_{i_{\tilde{z}}}$ through the points of $V_{\tilde{z}}$.

DEFINITION 5.5. *The Thomas tube associated with V is the subset of the microbundle $\tilde{\mathfrak{N}}_0$,*

$$(20) \quad \tilde{\mathfrak{N}}(V) = \bigcup_{\tilde{z} \in \tilde{\mathcal{M}}_0} \tilde{\mathfrak{U}}_{\tilde{z}}^V \subset \tilde{\mathfrak{N}}_0.$$

The image $\Pi(\tilde{\mathfrak{N}}(V)) \subset \mathfrak{M}$ is the saturation by \mathcal{F} of the clopen set V , hence $\Pi(\tilde{\mathfrak{N}}(V)) = \mathfrak{M}$. Note that each leaf \tilde{L} of $\tilde{\mathcal{F}}$ in $\tilde{\mathfrak{N}}(V)$ has no holonomy and is properly embedded by construction, though the projection L of \tilde{L} in \mathfrak{M} is recurrent by minimality. We consider another example.

EXAMPLE 5.6. Let \mathfrak{M} be a weak solenoid of Rogers and Tollefson [53], i.e. the inverse limit

$$\mathfrak{M} = \varprojlim \{f: K \rightarrow K\},$$

where $f: K \rightarrow K$ is a 2-fold covering of the Klein bottle K . In this case \mathfrak{M} contains a single non-orientable leaf L_1 with non-trivial holonomy, homeomorphic to a 2-ended cylinder C modulo some identification. Every other leaf is without holonomy and is homeomorphic to C .

Let L_0 be a leaf without holonomy, and let $\mathcal{U} = \{U_i\}_{1 \leq i \leq m}$ be a finite cover of \mathfrak{M} such that the transversals V_i are permuted under the action of $\mathcal{G}_{\mathcal{F}}^*$, that is, if $V_i \subset D(h_{\tilde{z}})$, $h_{\tilde{z}} \in \mathcal{G}_{\mathcal{F}}^*$, then there exists $V_j = h_{\tilde{z}}(V_i)$. Let \mathcal{M}_0 be a Delaunay set in L_0 . Then the foliated microbundle $\tilde{\mathfrak{N}}_0$ contains leaves homeomorphic to a 2-ended infinite cylinder, and each of these leaves projects under Π onto a dense leaf in \mathfrak{M} , although none of the leaves is dense in $\tilde{\mathfrak{N}}_0$.

REMARK 5.7. As indicated in Example 5.6 the microbundle $\tilde{\mathfrak{N}}_0$ depends on the choice of a foliated atlas of \mathfrak{M} . However, if \mathcal{U}_1 and \mathcal{U}_2 are two regular covers, and \mathfrak{N}_0^1 and \mathfrak{N}_0^2 are corresponding microbundles constructed over the same leaf L_0 , there exists a constant $\epsilon < \min\{\epsilon_{\mathcal{U}_1}^{\mathcal{F}}, \epsilon_{\mathcal{U}_2}^{\mathcal{F}}\}$ such that foliations on \mathfrak{N}_0^1 and \mathfrak{N}_0^2 coincide if restricted to a neighborhood of \tilde{L}_0 , determined by ϵ .

5.4. Reeb neighborhoods of compact sets. Next we consider a modified form of the microbundle construction which applies to compact subsets of leaves as in Definition 1.2, when \mathfrak{M} is any matchbox manifold, without imposing the equicontinuity assumption on the dynamics.

Let $L_0 \subset \mathfrak{M}$ be a leaf, and $K \subset L_0$ a proper base, which is assumed to be a union of closed plaques in the foliation chart domains $\{U_i \mid 1 \leq i \leq \nu\}$. Then by assumption, there exists $\tilde{K} \subset \tilde{L}_0$ which is a connected compact subset of the holonomy covering $\Pi: \tilde{L}_0 \rightarrow L_0$, such that $K = \Pi(\tilde{K})$.

Let \mathcal{M}_0 be an (e_1, e_2) -net in L_0 , and $\tilde{\mathcal{M}}_0$ the lift to an (e_1, e_2) -net in \tilde{L}_0 . Then the set $\tilde{K} \cap \tilde{\mathcal{M}}_0$ is finite, and the image $\mathcal{M}_K = \Pi(\tilde{K} \cap \tilde{\mathcal{M}}) \subset \mathfrak{T}_*$ is a finite set of points. Choose a basepoint $\tilde{z}_0 \in (\tilde{K} \cap \tilde{\mathcal{M}}_0)$ and set $z_0 = \Pi(\tilde{z}_0)$.

DEFINITION 5.8. *A clopen neighborhood $z_0 \in V_{z_0} \subset \mathfrak{T}_*$ is \tilde{K} -admissible if $V_{z_0} \subset D(h_{\tilde{z}})$ for each $\tilde{z} \in \tilde{K} \cap \tilde{\mathcal{M}}_0$. In this case, the Reeb neighborhood of \tilde{K} is defined by*

$$(21) \quad \tilde{\mathfrak{N}}(\tilde{K}, V_{z_0}) \equiv \bigcup_{\tilde{z} \in \tilde{K} \cap \tilde{\mathcal{M}}_0} \tilde{\pi}_{\tilde{z}}^{-1}(\tilde{V}_{\tilde{z}}) \subset \tilde{\mathfrak{N}}_0.$$

If V_{z_0} is \tilde{K} -admissible and the restricted map $\Pi: \tilde{\mathfrak{N}}(\tilde{K}, V_{z_0}) \rightarrow \mathfrak{M}$ is injective, then we say that V_{z_0} is \tilde{K} -disjoint. In particular, the collection of images $\{V_{\tilde{z}} \equiv h_{\tilde{z}}(V_{z_0}) \mid \tilde{z} \in \tilde{K} \cap \tilde{\mathcal{M}}_0\}$ form a disjoint subset of \mathfrak{T}_ . Also, define*

$$(22) \quad \mathfrak{N}(K, V_{z_0}) \equiv \Pi(\tilde{\mathfrak{N}}(\tilde{K}, V_{z_0})) = \bigcup_{\tilde{z} \in \tilde{K} \cap \tilde{\mathcal{M}}_0} \Pi\{\tilde{\pi}_{\tilde{z}}^{-1}(\tilde{V}_{\tilde{z}})\} \subset \mathfrak{M}.$$

Note that each leaf of the restricted foliation $\tilde{\mathcal{F}}|_{\tilde{\mathfrak{N}}(\tilde{K}, V_{z_0})}$ is a properly embedded compact subset, and the holonomy $\tilde{h}_{\tilde{\gamma}}$ along any closed loop $\tilde{\gamma}$ contained in $\tilde{\mathfrak{N}}(\tilde{K}, V_{z_0})$ is trivial. If V_{z_0} is \tilde{K} -disjoint, then the same holds for each path component of $\mathfrak{N}(K, V_{z_0})$. On the other hand, the restricted foliation on the image $\Pi\{\tilde{\mathfrak{N}}(\tilde{K}, V_{z_0})\}$ will have recurrence unless V_{z_0} is \tilde{K} -disjoint.

6. TRANSVERSE CANTOR FOLIATIONS

For some classes of examples of matchbox manifolds, there is given as part of the defining data, a continuous family of local transversals to the foliation \mathcal{F} on \mathfrak{M} . These include the generalized solenoids, which are fibrations with base an n -dimensional manifold and Cantor space fibers, and the suspension of a topological action of a group on a Cantor set, which also yields a Cantor fibration. For the tiling spaces associated to aperiodic tilings with finite local complexity of \mathbb{R}^n , Sadun and Williams showed that their tiling spaces are homeomorphic to a Cantor fibration [54], though the conjugation does not preserve the natural \mathbb{R}^n -action on leaves. For the more general construction of tilings spaces in [12] associated to a homogeneous space $N = G/K$ defined by an action of a connected Lie group G , the tiling space Ω_G is homeomorphic to an expansive minimal G -solenoid, but it is not known in what generality they admit a fibration structure. Thus, in some cases, the associated matchbox manifold \mathfrak{M} is a fibration over a base space with fibers homeomorphic to a Cantor set, while in other cases the local transversals define local fibrations, but not a global fibration structure.

In this section, we introduce the notion of a *Cantor foliation* \mathcal{H} on a closed subset $\mathfrak{B} \subset \mathfrak{M}$, which formalizes the properties of the continuous family of local transversals in the above examples. Since a Cantor set is totally disconnected, the leaves of a Cantor foliation \mathcal{H} on \mathfrak{B} cannot simply be defined in terms of the connected components for some finer topology on \mathfrak{B} . In the works by Putnam [49, 50], for example, the “leaves” of \mathcal{H} are defined dynamically, as stable or unstable manifolds for a “hyperbolic” action on a Cantor set. In the context of matchbox manifolds, we use the regularity imposed by a covering by coordinate charts to define the leaves. The definition then requires some constraints on the space \mathfrak{B} , as specified below, which possibly limits the variety of possible examples one might consider. However, our definition suffices for the examples discussed above, and for the conclusions of Theorems 1.1 and ??.

6.1. Cantor foliations. Recall that we assume there is a fixed regular covering $\{U_i \mid 1 \leq i \leq \nu\}$ of \mathfrak{M} by foliation charts, as in Proposition 2.6, with charts $\varphi_i: \overline{U}_i \rightarrow [-1, 1]^n \times \mathfrak{T}_i$ where $\mathfrak{T}_i \subset \mathfrak{T}$ is a clopen subset. By construction (see [20]), each chart admits a foliated extension $\hat{\varphi}_i: \hat{U}_i \rightarrow (-2, 2)^n \times \mathfrak{T}_i$ where $\overline{U}_i \subset \hat{U}_i \subset \mathfrak{M}$ is an open neighborhood of \overline{U}_i and $\hat{\varphi}_i|_{\overline{U}_i} = \varphi_i$. Following the model of the construction in (19), for a clopen set $V \subset \mathfrak{T}_i$ set $\mathfrak{U}_i^V = \pi_i^{-1}(V) \subset \overline{U}_i$ and $\hat{\mathfrak{U}}_i^V = \hat{\varphi}_i^{-1}(V) \subset \hat{U}_i$.

DEFINITION 6.1. Let \mathfrak{M} be a matchbox manifold, and $\mathfrak{B} \subset \mathfrak{M}$ a closed subset. An equivalence relation $\approx_{\mathcal{H}}$ on \mathfrak{B} is said to define a transverse Cantor foliation \mathcal{H} of \mathfrak{B} if for each $x \in \mathfrak{B}$, the class $\mathcal{H}_x = \{y \in \mathfrak{B} \mid y \approx_{\mathcal{H}} x\}$ is a Cantor set. Moreover, we require that there exists a covering of \mathfrak{M} by foliation charts as above, such that for each $x \in \mathfrak{B}$, there exists:

- (1) $1 \leq i_x \leq \nu$ with $x \in U_{i_x}$,
- (2) a clopen subset $V_x \subset \mathfrak{T}_{i_x}$ with $w_x = \pi_{i_x}(x) \in V_x$ and $\mathfrak{U}_{i_x}^{V_x} \subset \mathfrak{B}$;
- (3) a homeomorphism into $\Phi_x: [-1, 1]^n \times V_x \rightarrow \widehat{U}_{i_x}$ such that

$$\Phi_x(\xi, w_x) = \widehat{\varphi}^{-1}(\xi, w_x) \text{ for } \xi \in [-1, 1]^n,$$
- (4) for $\xi \in [-1, 1]^n$ and $z = \widehat{\varphi}^{-1}(\xi, w_x)$, the image $\Phi_x(\{\xi\} \times V_x) = \mathcal{H}_z \cap \widehat{\mathfrak{U}}_{i_x}^{V_x}$.

The leaves of the “foliation” \mathcal{H} are defined to be the equivalence classes \mathcal{H}_x of $\approx_{\mathcal{H}}$ in \mathfrak{B} .

Conditions 6.1.3 and 6.1.4 imply that the leaves of the Cantor foliation are “vertical” segments for a regular coordinate chart, after reparametrization by the maps Φ_x . The functions Φ_x are the adjustments to the local vertical foliation defined by the foliation coordinate system, so that the leaves of \mathcal{H} defined locally in a chart are coordinate independent, hence are well defined on \mathfrak{B} . The images of the maps Φ_x are allowed to take values in the open neighborhood \widehat{U}_{i_x} as the “leaf” \mathcal{H}_z may not have constant horizontal coordinate λ_{i_x} . In particular, for a leafwise boundary point $z \in \mathfrak{B} \cap \overline{U}_{i_x}$, the equivalence class \mathcal{H}_z need not be contained in \overline{U}_{i_x} .

The functions Φ_x specified by Conditions 6.1.3 and 6.1.4 define a new set of foliation coordinate charts for \mathfrak{M} , for which the transverse Cantor foliation \mathcal{H} is given by the vertical coordinate directions. However, the functions Φ_x are not required to be leafwise smooth, so that these redefined charts only yield the structure of a topological manifold on leaves. This is the case for the constructions below, which yield functions Φ_x which are piecewise-linear maps when restricted to leaves, but are not smooth by construction, though they may be smoothable.

If the functions Φ_x specified by Conditions 6.1.3 and 6.1.4 are leafwise smooth, then these redefined charts will also be leafwise smooth, so that we are in the situation of the examples cited above.

6.2. Cantor foliations and microbundles. Next, consider a leaf $L_0 \subset \mathfrak{M}$ with holonomy covering $\Pi: \widetilde{L}_0 \rightarrow L_0$. Let \mathcal{M}_0 be an (e_1, e_2) -net for L_0 chosen as in Section 5.1, and $\widetilde{\mathcal{M}}_0$ the net on \widetilde{L}_0 defined by the covering map $\Pi: \widetilde{L}_0 \rightarrow L_0$. Then Definition 6.1 of a transverse Cantor foliation extends verbatim to the context of a compact subset $\mathfrak{B} \subset \mathfrak{N}_0$ of the foliated microbundle over \widetilde{L}_0 .

Let $K \subset L_0$ be a proper base. Chose a basepoint $z_0 \in K \cap \mathcal{M}_0$, with lift $\widetilde{z}_0 \in \widetilde{K} \cap \widetilde{L}_0$. Let $V_{z_0} \subset \mathfrak{T}_0$ be a clopen subset such that $\mathfrak{N}(K, V_{z_0}) \subset \mathfrak{N}_0$ is defined by (21). We consider the special properties of a transverse Cantor foliation on $\mathfrak{B} = \mathfrak{N}(K, V_{z_0})$ which is modeled on $V_{\widetilde{z}}$ in the chart $\mathfrak{U}_{\widetilde{z}}^{V_{\widetilde{z}}}$ as defined in (19). That is, for $\widetilde{y} \in \mathfrak{U}_{\widetilde{z}}^{V_{\widetilde{z}}}$ the projection of the equivalence class $\widetilde{\mathcal{H}}_{\widetilde{y}} \subset \mathfrak{N}(K, V_{z_0})$ to $\mathfrak{T}_{\widetilde{z}}$ is a homeomorphism onto. Let $\widetilde{\Phi}_{\widetilde{z}}$ be the map defined by (6.1.3) for the chart $\mathfrak{U}_{\widetilde{z}}^{V_{\widetilde{z}}}$ and then define

$$(23) \quad \mathfrak{U}(\widetilde{K}, V_{z_0}) \equiv \bigcup_{\widetilde{z} \in \widetilde{K} \cap \widetilde{\mathcal{M}}_0} \widetilde{\Phi}_{\widetilde{z}}([-1, 1]^n \times V_{\widetilde{z}}).$$

LEMMA 6.2. The set $\mathfrak{U}(\widetilde{K}, V_{z_0})$ is a bi-foliated neighborhood of \widetilde{K} in the foliated microbundle \mathfrak{N}_0 for which there exists a bi-foliated homeomorphism

$$(24) \quad \widetilde{\Phi}: \mathfrak{U}(\widetilde{K}, V_{z_0}) \rightarrow \widetilde{K} \times V_{z_0}.$$

Proof. Condition 6.1.1 implies that the Cantor foliation $\widetilde{\mathcal{H}}$ on $\mathfrak{U}(\widetilde{K}, V_{z_0})$ defines a projection map $p_{\widetilde{K}}: \mathfrak{U}(\widetilde{K}, V_{z_0}) \rightarrow \widetilde{K}$. Given $\widetilde{x} \in \widetilde{K}$ and $\widetilde{y} \in \widetilde{\mathcal{H}}_{\widetilde{x}}$ the leaf $\widetilde{L}_{\widetilde{y}}$ intersects the transversal $\widetilde{\mathcal{H}}_{\widetilde{z}_0} \cong V_{z_0}$ in a unique point $w_{\widetilde{y}}$. We then set $\widetilde{\Phi}(\widetilde{y}) = (\widetilde{x}, w_{\widetilde{y}}) \in \widetilde{K} \times V_{z_0}$. \square

DEFINITION 6.3. A transverse Cantor foliation $\widetilde{\mathcal{H}}$ for a compact subset $\widetilde{\mathfrak{B}} \subset \mathfrak{N}_0$ is said to holonomy equivariant if it defines a transverse Cantor foliation \mathcal{H} on the image $\mathfrak{B} = \Pi(\widetilde{\mathfrak{B}}) \subset \mathfrak{M}$.

If the restriction $\Pi: \tilde{\mathfrak{B}} \rightarrow \mathfrak{M}$ is injective, then this is always the case. For example, if V_{z_0} is \tilde{K} -disjoint then any transverse Cantor foliation $\tilde{\mathcal{H}}$ on $\mathfrak{U}(\tilde{K}, V_{z_0})$ will be holonomy equivariant. On the other hand, if V_{z_0} is \tilde{K} -admissible but not \tilde{K} -disjoint, then $\tilde{\mathcal{H}}$ is holonomy equivariant if the images of the equivalence classes $\Pi(\tilde{\mathcal{H}}_{\tilde{y}})$ agree on the overlap of any two coordinate charts on $\mathfrak{U}(\tilde{K}, V_{z_0})$. This condition will be satisfied, for example, if the equivalence classes $\tilde{\mathcal{H}}_{\tilde{x}}$ for $\tilde{x} \in \tilde{K}$ are defined in terms of a transverse Cantor foliation \mathcal{H} on the image $\Pi(\mathfrak{U}(\tilde{K}, V_{z_0})) \subset \mathfrak{M}$.

DEFINITION 6.4. *A pair $\{\tilde{K}, V_{z_0}\}$ is a complete model for \mathfrak{M} , if V_{z_0} is \tilde{K} -admissible, and the map $\Pi: \tilde{\mathfrak{N}}(\tilde{K}, V_{z_0}) \rightarrow \mathfrak{M}$ is surjective.*

The basic observation is that if $\{\tilde{K}, V_{z_0}\}$ is a complete model for \mathfrak{M} , and $\tilde{\mathcal{H}}$ is a holonomy equivariant transverse Cantor foliation for $\tilde{\mathfrak{B}} = \tilde{\mathfrak{N}}(\tilde{K}, V_{z_0})$, then $\tilde{\mathcal{H}}$ defines a transverse Cantor foliation on \mathfrak{M} .

LEMMA 6.5. *Let \mathfrak{M} be an equicontinuous matchbox manifold. Then there exists $\{\tilde{K}, V_{z_0}\}$ which is a complete model for \mathfrak{M} .*

Proof. The foliation \mathcal{F} of \mathfrak{M} is minimal by Theorem 4.7. Choose a $\mathcal{G}_{\mathcal{F}}$ -invariant clopen subset $w_0 \in V \subset \mathfrak{T}_*$ as in Section 5.3. Then for any leaf $L_0 \subset \mathfrak{M}$, the set V is \tilde{K} -admissible for all $\tilde{K} \subset \tilde{L}_0$. As the leaf L_0 is dense, and V is clopen, for R sufficiently large we can take $\tilde{K} = D_{\tilde{L}_0}(\tilde{z}_0, R)$ and obtain a complete model $\{D_{\tilde{L}_0}(\tilde{z}_0, R), V\}$. \square

PART II - FOLIATED VORONOI AND DELAUNAY STRUCTURES

The concept of a Voronoi cell decomposition (or tessellation) of the plane, or of a Euclidean space more generally, is extraordinarily useful for applications of geometry to a variety of problems, and is very well-studied. For a nice historical discussion of this concept, and a good description of such applications, see the *Introduction* of the book [48]. In the following Sections 7 and 8, we develop the basic concepts of Voronoi tessellations and Delaunay triangulations, especially in a form applicable to metric spaces derived from the leaves of matchbox manifolds. Section 9 applies these techniques to construct transverse Cantor foliations for closed sets \mathfrak{B} in \mathfrak{M} .

Recall we assume there is a fixed regular covering $\{U_i \mid 1 \leq i \leq \nu\}$ of \mathfrak{M} by foliation charts, as in Proposition 2.6, with charts $\varphi_i: \overline{U}_i \rightarrow [-1, 1]^n \times \mathfrak{T}_i$ where each $\mathfrak{T}_i \subset \mathfrak{X}$ is a clopen subset.

7. VORONOI TESSELLATIONS

Let \mathfrak{M} be a matchbox manifold, and $L \subset \mathfrak{M}$ a leaf with induced leafwise Riemannian metric d_L . Let $X \subset L$ be a closed connected set which is a union of plaques. The typical examples we consider are for $X = L$, or for X a compact subset of L which contains a proper base K in its interior.

Recall that $\lambda_{\mathcal{F}} > 0$ is the leafwise constant defined in Lemma 2.5, such that for all $x \in L$, the closed disk $D_L(x, \lambda_{\mathcal{F}}) \subset L$ is strongly convex.

Let \mathcal{N}_X be a given (d_1, d_2) -net for X . Recall from Definition 5.1, that this means that there are constants $0 < d_1 < d_2$, such that for all $y \neq z \in \mathcal{N}_X$, $d_1 \leq d_L(y, z)$, i.e. \mathcal{N}_X is d_1 -separated, and for all $x \in X$, there exists some $z \in \mathcal{N}_X$ such that $d_L(x, z) \leq d_2$, i.e. \mathcal{N}_X is d_2 -dense in X .

Note that the (d_1, d_2) -net \mathcal{N}_X is defined independently of the choice of the net \mathcal{M}_0 chosen in Section 5.1. We always assume that $d_2 \leq \lambda_{\mathcal{F}}/5$, though as chosen later in this work, d_2 is typically much smaller than the density constant $e_2 = \epsilon_{\mathcal{U}}^{\mathcal{F}}/4$ for \mathcal{M}_0 from Section 5.1.

Associated to the net \mathcal{N}_X is the *Voronoi tessellation*, which is a partition of the space into compact star-like regions, called cells. The cells can be thought of as a tiling of X , although there is no assumption that there is only a finite number of isometry types of cells. Thus, we are considering the most general form of a tiling.

7.1. Voronoi cells. Introduce the “leafwise nearest-neighbor distance” function, where for $y \in L$,

$$(25) \quad \kappa_X(y) = \inf \{d_L(x, y) \mid x \in \mathcal{N}_X\}.$$

Note that $\kappa_X(y) = 0$ if and only if $y \in \mathcal{N}_X$.

DEFINITION 7.1. For $x \in \mathcal{N}_X$, define its Dirichlet region, or Voronoi cell, in L by

$$(26) \quad \mathcal{C}(x) = \{y \in L \mid d_L(x, y) = \kappa_X(y)\}.$$

That is, for $x \in \mathcal{N}_X$ the Voronoi cell $\mathcal{C}(x)$ consists of the points $y \in L$ which are closer to x in the leafwise metric than to any other point of \mathcal{N}_X . Thus, for each $y \in L$ there exists some $x \in \mathcal{N}_X$ with $y \in \mathcal{C}(x)$. In particular, for $\mathcal{C}_X(x) = \mathcal{C}(x) \cap X$, then the collection of closed subsets

$$(27) \quad \{\mathcal{C}_X(x) \mid x \in \mathcal{N}_X\}$$

forms a closed covering of X , and we obtain the Voronoi decomposition of X

$$(28) \quad X = \bigcup_{x \in \mathcal{N}_X} \mathcal{C}_X(x)$$

Introduce the subset of \mathcal{N}_X consisting of net points whose Voronoi cells lie in X ,

$$(29) \quad \mathcal{N}_X^* = \{x \in \mathcal{N}_X \mid \mathcal{C}(x) \subset X\}$$

We develop some of the properties of the cells $\mathcal{C}(x)$ for $x \in \mathcal{N}_X$ and \mathcal{N}_X^* . In particular, Lemma 7.5 below and the assumptions that X is a union of plaques and that $d_2 \leq e_2$ implies \mathcal{N}_X^* is not empty.

LEMMA 7.2. For each $x \in \mathcal{N}_X^*$,

$$(30) \quad D_L(x, d_1/2) \subset \mathcal{C}(x) \subset D_L(x, d_2)$$

In particular, $\mathcal{C}(x)$ has diameter at most $2d_2$.

Proof. Note that for all $y \in X$, we have $\kappa_X(y) \leq d_2$ as there exists $z \in \mathcal{N}_X$ such that $d_L(y, z) \leq d_2$. Hence, for $x \in \mathcal{N}_X^*$, we are given that $\mathcal{C}(x) \subset X$, so $\mathcal{C}(x) \subset D_L(x, d_2)$.

On the other hand, for all $x \neq y \in \mathcal{N}_X$ we have $x \notin B_L(y, d_1)$ and thus $B_L(x, d_1/2) \cap B_L(y, d_1/2) = \emptyset$. The inclusion $D_L(x, d_1/2) \subset \mathcal{C}_X(x)$ follows as a result. \square

The upper bound estimate in (30) need not hold if $\mathcal{C}(x) \not\subset X$, for then the set \mathcal{N}_X is not d_2 -dense in all of L . However, we always have:

LEMMA 7.3. For $x \in \mathcal{N}_X$, $\mathcal{C}_X(x) \subset D_L(x, d_2)$.

Proof. Let $y \in \mathcal{C}(x) \cap X$. As \mathcal{N}_X is d_2 -dense in X , there exists $z \in \mathcal{N}_X$ with $d_L(y, z) \leq d_2$. As $y \in \mathcal{C}(x)$, $x \in \mathcal{N}_X$ is the closest net point, thus $d_L(y, x) \leq d_L(y, z) \leq d_2$. \square

A set $Y \subset L$ is *star-like with respect to* $x \in Y$ if for all $y \in Y$, each geodesic ray from x to y is contained in Y .

LEMMA 7.4. For each $x \in \mathcal{N}_X$, the set $Y = \mathcal{C}(x) \cap D_L(x, \lambda_{\mathcal{F}})$ is star-like with respect to x . In particular, for all $x \in \mathcal{N}_X^*$ the set $\mathcal{C}(x)$ is star-like with respect to x .

Proof. For $y \in \mathcal{C}(x) \cap D_L(x, \lambda_{\mathcal{F}})$ there is a unique geodesic segment $\sigma_{x,y}: [0, 1] \rightarrow D_L(x, \lambda_{\mathcal{F}}) \subset L$ with $\sigma_{x,y}(0) = x$ and $\sigma_{x,y}(1) = y$. Let $z = \sigma_{x,y}(s)$ for $0 < s < 1$, then we show that $z \in \mathcal{C}(x)$.

Let $u \in \mathcal{N}_X$ with $u \neq x$, then $d_L(y, x) \leq d_L(y, u)$. The strong convexity of $D_L(x, \lambda_{\mathcal{F}})$ implies that $d_L(x, z) = s \cdot d_L(x, y)$ and $d_L(z, y) = (1 - s) \cdot d_L(x, y)$. Then the triangle inequality implies that

$$d_L(z, u) \geq d_L(y, u) - d_L(z, y) = d_L(y, u) - (1 - s) \cdot d_L(x, y) \geq d_L(y, x) - (1 - s) \cdot d_L(x, y)$$

so $d_L(x, z) \leq d_L(u, z)$. Thus, $z \in \mathcal{C}(x)$. \square

The strong convexity of disks $D_L(x, \lambda_{\mathcal{F}})$ also yields the following.

LEMMA 7.5. *If $x \in \mathcal{N}_X$ and there exists $r > d_2$ for which $B_L(x, r) \subset X$, then $x \in \mathcal{N}_X^*$.*

Proof. Suppose that $y \in \mathcal{C}(x)$ but $y \notin X$. Let $\sigma_{x,y}: [0, 1] \rightarrow L$ be a geodesic segment with $\sigma_{x,y}(0) = x$ and $\sigma_{x,y}(1) = y$, and the length equal to $d_L(x, y)$. Let $0 < s < 1$ be the greatest value for which $y' = \sigma_{x,y}(s) \in \mathcal{C}_X(x)$, then $d_2(x, y') \geq r > d_2$ by assumption. As $y' \in X$, there exists $z \in \mathcal{N}_X$ with $d_L(y', z) < d_2$. Then

$$d_L(y, z) \leq d_L(y, y') + d_L(y', z) < d_L(y, y') + d_2 < d_L(y, y') + r \leq d_L(y, y') + d_L(y', x) = d_L(y, x)$$

which contradicts that $y \in \mathcal{C}(x)$. Thus, $\mathcal{C}(x) \subset X$ hence $x \in \mathcal{N}_X^*$. \square

Next, we introduce the *star-neighborhoods* of Voronoi cells. Given $x \in \mathcal{N}_X$, introduce the *vertex-sets*

$$(31) \quad \mathcal{V}_X(x) = \{y \in \mathcal{N}_X \mid \mathcal{C}_X(y) \cap \mathcal{C}_X(x) \neq \emptyset\}; \quad \mathcal{V}_X^*(x) = \{y \in \mathcal{V}_X(x) \mid y \neq x\}.$$

Note that $\mathcal{V}_X(x)$ is a finite set by the net condition on \mathcal{N}_X , and $y \in \mathcal{V}_X^*(x)$ if and only if $x \in \mathcal{V}_X^*(y)$.

DEFINITION 7.6. *For $x \in \mathcal{N}_X$ the “star-neighborhood” of the Voronoi cell $\mathcal{C}_X(x)$ is the set*

$$(32) \quad \mathcal{S}_X(x) = \bigcup_{y \in \mathcal{V}_X(x)} \mathcal{C}_X(y).$$

LEMMA 7.7. *Assume that $d_2 \leq \lambda_{\mathcal{F}}/5$. For each $x \in \mathcal{N}_X$, $\mathcal{S}_X(x) \subset B_L(x, 3d_2) \subset B_L(x, \lambda_{\mathcal{F}})$, hence $\mathcal{S}_X(x)$ is contained in a strongly convex subset of L .*

Proof. Suppose that $\mathcal{C}_X(x) \cap \mathcal{C}_X(y) \neq \emptyset$. As $\mathcal{C}_X(z)$ has diameter at most $2d_2$ for all $z \in \mathcal{N}_X$ by Lemma 7.3, we obtain $\mathcal{S}_X(x) \subset B_L(x, 3d_2)$. As $d_2 \leq \lambda_{\mathcal{F}}/5$, the claim follows. \square

For $y \in \mathcal{V}_X^*(x)$ set

$$(33) \quad H(x, y) = \{z \in D_L(x, \lambda_{\mathcal{F}}) \mid d_L(x, z) \leq d_L(y, z)\}.$$

Thus $H(x, y)$ contains the set of points in the closed disk $D_L(x, \lambda_{\mathcal{F}})$ which are closer to x than to y .

Clearly, each $H(x, y)$ is closed, and the strong convexity of $D_L(x, \lambda_{\mathcal{F}})$ implies that for $x \in \mathcal{N}_X^*$,

$$(34) \quad \mathcal{C}(x) = \bigcap_{y \in \mathcal{V}_X^*(x)} H(x, y).$$

Conversely, $\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset$ implies that the intersection

$$(35) \quad L(x, y) = H(x, y) \cap H(y, x) \neq \emptyset.$$

For example, if L is isometric to Euclidean space \mathbb{R}^2 , then $H(x, y)$ is the intersection of the disk $D_L(x, \lambda_{\mathcal{F}})$ with the half-plane in \mathbb{R}^2 consisting of the points which are closer to x than to y . Thus, $L(x, y)$ is a line segment. In the more general case, where L is a complete Riemannian manifold, then the local picture of $L(x, y)$ is similar to the Euclidean case, as seen below. However, unless L has a global convexity property, L may have focal points and the global structure of $L(x, y)$ is not so easily described. Thus, we restrict consideration to convex neighborhoods.

LEMMA 7.8. *For $x \in \mathcal{N}_X$ and $y \in \mathcal{V}_X^*(x)$, $L(x, y) \cap D_L(x, \lambda_{\mathcal{F}})$ is a codimension-one closed submanifold.*

Proof. We have $x, y \in D_L(x, \lambda_{\mathcal{F}})$ by Lemma 7.7. As the metric d_L is strongly convex when restricted to $D_L(x, \lambda_{\mathcal{F}})$, the functions $f_x(z) = d_L(x, z)^2$ and $f_y(z) = d_L(y, z)^2$ are both regular on $D_L(x, \lambda_{\mathcal{F}})$, which implies that $L(x, y) \cap D_L(x, \lambda_{\mathcal{F}})$ is a codimension-one closed submanifold. \square

Now restrict attention to $x \in \mathcal{N}_X^*$ so that $\mathcal{C}(x) \subset X$. Let $\mathcal{V}_X^1(x) \subset \mathcal{V}_X^*(x)$ be the subset corresponding to the codimension-one faces of the boundary of $\mathcal{C}(x)$. That is, $y \in \mathcal{V}_X^1(x)$ if and only if $\partial_y \mathcal{C}_X(x) = \mathcal{C}(x) \cap L(x, y)$ has non-trivial interior as a subset of the submanifold $L(x, y)$. Then the topological boundary $\partial \mathcal{C}_X(x)$ is the finite union

$$(36) \quad \partial \mathcal{C}_X(x) = \bigcup_{y \in \mathcal{V}_X^1(x)} \partial_y \mathcal{C}_X(x).$$

We summarize the results of this section.

PROPOSITION 7.9. *Let \mathcal{N}_X be an (d_1, d_2) -net in X , such that $d_2 \leq \lambda_{\mathcal{F}}/5$. Then there exists a subset $\mathcal{N}_X^* \subset \mathcal{N}_X$ and a collection of closed sets $\{\mathcal{C}_X(y) \mid y \in \mathcal{N}_X\}$ satisfying:*

- (1) $\mathcal{C}_X(x) \subset X$ for each $x \in \mathcal{N}_X$;
- (2) $\mathcal{C}_X(x) \subset D_L(x, d_2)$ for each $x \in \mathcal{N}_X$;
- (3) $\text{int}(\mathcal{C}_X(x)) \cap \text{int}(\mathcal{C}_X(y)) = \emptyset$ for each pair $x \neq y \in \mathcal{N}_X$;
- (4) The collection $\{\mathcal{C}_X(y) \mid y \in \mathcal{N}_X\}$ is a closed covering of X .

In addition, for $x \in \mathcal{N}_X^*$ we have:

- (5) $\mathcal{C}_X(x) = \mathcal{C}(x)$;
- (6) $D_L(x, d_1/2) \subset \mathcal{C}_X(x)$;
- (7) $\mathcal{C}_X(x)$ is star-like with respect to x ;
- (8) $\partial\mathcal{C}_X(x)$ is a union of codimension-one submanifolds with boundary.

The collection $\{\mathcal{C}_X(y) \mid y \in \mathcal{N}_X\}$ is called the *Voronoi tessellation* of X associated to \mathcal{N}_X .

8. DELAUNAY SIMPLICIAL COMPLEX

We next introduce the *Delaunay simplicial complex* obtained from a (d_1, d_2) -net \mathcal{N}_X for $X \subset L$, using the “inscribed sphere” characterization of the simplices. We assume that $d_2 \leq \lambda_{\mathcal{F}}/5$ as always.

Let $0 < r < \lambda_{\mathcal{F}}$. Then the leafwise sphere of radius r centered at z is

$$S_L(z, r) \equiv \{y \in L \mid d_L(z, y) = r\} = D_L(z, r) - B_L(z, r).$$

Note that if $B_L(x, r) \cap \mathcal{N}_X = \emptyset$ for $x \in X$, then $r < d_2$ by the definition of d_2 .

8.1. Definition of a simplicial complex. The *Delaunay complex* $\Delta(\mathcal{N}_X)$ of L derived from the net \mathcal{N}_X is defined by specifying the subsets of \mathcal{N}_X which form the vertices of the simplices in $\Delta(\mathcal{N}_X)$. For $k \geq 0$, denote by $\Delta^{(k)}(\mathcal{N}_X)$ the collection of k -simplices, defined as follows:

DEFINITION 8.1. *For each $z_0 \in \mathcal{N}_X$, the set $\Delta(z_0) = \{z_0\}$ is a 0-simplex in $\Delta^{(0)}(\mathcal{N}_X)$.*

For $k > 0$, a $(k+1)$ -tuple $\{z_0, \dots, z_k\} \subset \mathcal{N}_X$ forms a k -simplex $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{N}_X)$ if there exists $x \in L$ and $0 < r \leq d_2$ such that $B_L(x, r) \cap \mathcal{N}_X = \emptyset$, and $\{z_0, \dots, z_k\} \subset S_L(x, r) \cap \mathcal{N}_X$. Then $S_L(x, r)$ is called the inscribed sphere of the simplex $\{z_0, \dots, z_k\}$.

If $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{N}_X)$, then every subset of $(\ell+1)$ -points, $\{z_{i_0}, \dots, z_{i_\ell}\} \subset \{z_0, \dots, z_k\}$ yields an ℓ -simplex $\Delta(z_{i_0}, \dots, z_{i_\ell}) \in \Delta^{(\ell)}(\mathcal{N}_X)$, as the inscribed sphere condition holds for all subsets. In particular, we have well-defined face and boundary operators defined on $\Delta(\mathcal{N}_X)$.

8.2. Realization of a Delaunay simplex. If the manifold L is Euclidean, then given a k -simplex $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{N}_X)$, the convex hull of the vertices defines a geometric k -simplex in L , which is its *geometric realization*. For a non-Euclidean manifold, this elementary and intuitive approach need not work, as the convex span of a k -simplex need not be a k -dimensional subset if the leaves have curvature. Rather, one must choose a procedure for “filling in” the geometric simplex spanned by a set of vertices, in order to obtain a geometric realization.

For a 1-simplex $\Delta(z_0, z_1)$, there is a canonical “filling in” using the geodesic between z_1 and z_0 , which is unique due to the strong convexity of $B_L(z_0, \lambda_{\mathcal{F}})$. For higher-dimensional simplices, we use an inductive procedure to fill in the faces using the geodesic cone from each successive vertex.

Define the standard k -simplex Δ^k in \mathbb{R}^{k+1} by the barycentric coordinate approach,

$$\Delta^k = \{(t_0, \dots, t_k) \mid t_\ell \geq 0, t_0 + \dots + t_k = 1\}.$$

The vertices of Δ^k are the coordinate vectors $\vec{e}_\ell = (0, \dots, 1, \dots, 0)$ where the unique non-zero entry is in the $(\ell+1)$ -coordinate position.

LEMMA 8.2. *Let $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{N}_X)$ be given, so that $\{z_0, \dots, z_k\} \subset B_L(z_0, \lambda_{\mathcal{F}})$. Then there exists a diffeomorphism $\sigma_k: \Delta^k \rightarrow L$ such that $\sigma_k(\vec{e}_\ell) = z_\ell$, and the maps $\{\sigma_i \mid 0 \leq i \leq k\}$ are natural with respect to the face operators.*

Proof. The map σ_k is defined by induction on the dimensions of the faces of Δ^k . Set $\sigma_k(\vec{e}_\ell) = z_\ell$.

Given a string $I = i_0 < i_1 < \dots < i_\nu$ with $0 \leq i_0$ and $i_\nu \leq k$, define the I -face $\partial_I \Delta^k$ to be the subset consisting of points where the only non-zero entries are in the coordinates appearing in the string. For $\nu > 0$, let $I' = i_0 < i_1 < \dots < i_{\nu-1}$. By induction, we may assume that the map $\sigma_k: \partial_{I'} \Delta^k \rightarrow L$ has been defined.

Note that each point $\vec{v} \in \partial_I \Delta^k$ can be written $\vec{v} = (1-s) \cdot \vec{v}' + s \cdot \vec{e}_{i_\nu}$ where $\vec{v}' \in \partial_{I'} \Delta^k$ and $0 \leq s \leq 1$. The point $z' = \sigma_k(\vec{v}') \in L$ is defined by the inductive hypothesis, and so there exists a unique geodesic segment $\tau: [0, 1] \rightarrow B_L(z_0, \lambda_{\mathcal{F}})$ such that $\tau(0) = z'$ and $\tau(1) = z_{i_\nu}$. Then set $\sigma_k(\vec{v}) = \tau(s)$. The resulting map defined on Δ^k satisfies the conclusions of Lemma 8.2. \square

DEFINITION 8.3. *Let $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{N}_X)$, then the geometric realization is the set*

$$(37) \quad |\Delta(z_0, \dots, z_k)| = \sigma_k(\Delta^k)$$

LEMMA 8.4. *For all $0 \leq \ell \leq k$, we have $|\Delta(z_0, \dots, z_k)| \subset B_L(z_\ell, \lambda_{\mathcal{F}})$.*

Proof. Let $x \in L$ and $0 < r \leq d_2$ such that $\{z_0, \dots, z_k\} \subset S_L(z, r) \cap \mathcal{N}_X$. Thus, $d_L(z_\ell, z_{\ell'}) \leq 2d_2$ for all $0 \leq \ell' \leq k$, and so the set of vertices $\{z_0, \dots, z_k\} \subset D(z_\ell, 2d_2) \subset B_L(z_\ell, \lambda_{\mathcal{F}})$. As $B_L(z_\ell, \lambda_{\mathcal{F}})$ is strongly convex, the geodesic segment between any two vertices of $\Delta(z_0, \dots, z_k)$ is also contained in $B_L(z_\ell, \lambda_{\mathcal{F}})$. Then proceed inductively, following the construction of σ_k in the proof of Lemma 8.2, and it follows that the image of the map σ_k is also contained in $B_L(z_\ell, \lambda_{\mathcal{F}})$. \square

REMARK 8.5. As was already mentioned, if the manifold L is not flat, the map σ_k may depend on the ordering of the set of vertices $\{z_0, \dots, z_k\}$ for $k > 1$, except on the edges of a simplex Δ^k . Indeed, the ordering of vertices in the string $I = i_0 < i_1 < \dots < i_\nu$ defines a choice of geodesic spray from the vertex with the largest index i_ν to the vertices i_ℓ with $\ell < \nu$. As the points of $\sigma_k(\partial_I \Delta^k)$ are obtained by flowing along geodesic curves, a different ordering of vertices defines a different choice of spanning geodesic rays. In case when L is a surface, this simply results in different parametrizations of the set $|\Delta(z_0, z_1, z_2)|$, as the boundary is 1-dimensional and so well-defined. If L is not flat and has dimension $n > 2$, the image of a point $\vec{v} \in \Delta^k$ need not be the same under the maps defined by these choices. In our applications, there will be given a “local ordering” of the points in \mathcal{N}_X , which defines an ordering of the set of vertices of a given simplex, so that the geometric realization $|\Delta(z_0, \dots, z_k)|$ will be well-defined.

The Voronoi cell decomposition and Delaunay triangulation of L are closely related. For Euclidean space, one says that $\Delta(\mathcal{N}_X)$ is dual to the Voronoi tessellation. For the general case of a Riemannian manifold with bounded geometry, we have the following results.

PROPOSITION 8.6. *For $z_0 \in \mathcal{N}_X^*$, let $\{z_1, \dots, z_k\} \subset \mathcal{V}_X^1(z_0)$. Then*

$$(38) \quad L(z_0, z_1) \cap \dots \cap L(z_0, z_k) \cap \mathcal{C}(z_0) \neq \emptyset \iff \Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{N}_X).$$

Proof. Recall that for $z \in \mathcal{N}_X^*$, $\mathcal{C}(z) \subset D_L(z, \lambda_{\mathcal{F}})$. Recall also that for $z \in \mathcal{N}_X^*$ and $y \in \mathcal{V}_X^*(z)$, $L(z, y) \cap D_L(z, \lambda_{\mathcal{F}})$ is a smooth submanifold formed by the intersecting boundaries of the Voronoi cells $\mathcal{C}(z)$ and $\mathcal{C}(y)$, and $\mathcal{V}_X^1(z) \subset \mathcal{V}_X^*(z)$ is the subset corresponding to the codimension-one faces of the boundary of $\mathcal{C}_X(z)$.

Let $x \in L(z_0, z_1) \cap \dots \cap L(z_0, z_k) \cap \mathcal{C}(z_0)$ and set $r = d_L(x, z_0)$. Then $d_L(x, z_i) = d_L(x, z_0) = r$ for each $1 \leq i \leq k$, and thus $\{z_0, \dots, z_k\} \subset S_L(x, r)$. As each $z_i \in \mathcal{V}_X^1(z_0)$, we have $d_L(z_0, z_i) \leq 2d_2$ and hence $r \leq d_2$. By the definition of the Voronoi cells, $B_L(x, r) \cap \mathcal{N}_X = \emptyset$. Suppose not, then there exists $y \in B_L(x, r) \cap \mathcal{N}_X$ with $d_L(y, x) < r = d_L(z_i, x)$ for $0 \leq i \leq k$, and so $x \notin \mathcal{C}(z_0) \subset X$. This implies $\Delta(z_0, \dots, z_k) \in \Delta^{(k)}(\mathcal{N}_X)$.

Conversely, for $\{z_1, \dots, z_k\} \subset \mathcal{V}_X^1(z_0)$ with $\{z_0, \dots, z_k\} \subset S_L(x, r)$, then x is equidistant from each point z_i and so $x \in L(z_0, z_j)$ for all $1 \leq i \leq k$. Moreover, for all $0 \leq i \leq k$, $x \in \mathcal{C}(z_i)$ as $B_L(x, r) \cap \mathcal{N}_X = \emptyset$ implies there is no $z \in \mathcal{N}_X$ with $d_L(x, z) < d_L(x, z_i)$.

Thus, $L(z_0, z_1) \cap \dots \cap L(z_0, z_k) \cap \mathcal{C}(z_i) \neq \emptyset$. \square

A point $x \in \partial\mathcal{C}(z_0)$ is called *extremal* if the distance function $d_L(z_0, y)$ has a local maximum on $\mathcal{C}(z_0)$ at $y = x$. Let $z_0 \in \mathcal{N}_X^*$, so that $\mathcal{C}_X(z_0) = \mathcal{C}(z_0)$. For $z_i \in \mathcal{V}_X^1(z_0)$, the boundary component $\partial_{z_i}\mathcal{C}(z_0) = \mathcal{C}(z_0) \cap L(z_0, z_i)$ has codimension one. Thus, for $z_0 \in \mathcal{N}_X^*$, a point $x \in \partial\mathcal{C}(z_0)$ is extremal exactly when there is $\{z_1, \dots, z_n\} \subset \mathcal{V}_X^1(z_0)$ with

$$x = \omega(z_0, \dots, z_n) = L(z_0, z_1) \cap \dots \cap L(z_0, z_n) \cap \mathcal{C}(z_0),$$

and $\omega(z_0, \dots, z_n)$ is the center of an inscribed sphere containing $\{z_0, \dots, z_n\}$ with radius

$$r(z_0, \dots, z_n) = d_{\mathcal{F}}(z_\ell, \omega(z_0, \dots, z_n)), \quad 0 \leq \ell \leq n.$$

Now introduce the simplicial cone of $z_0 \in \mathcal{N}_X^*$

$$(39) \quad \mathcal{C}_\Delta(z) = \bigcup \left\{ |\Delta(z_0, \dots, z_n)| \mid \Delta(z_0, z_1, \dots, z_n) \in \Delta^{(n)}(\mathcal{N}_X) \right\} \subset B(z, \lambda_{\mathcal{F}})$$

PROPOSITION 8.7. *For all $z \in \mathcal{N}_X^*$, $\mathcal{C}(z) \subset \mathcal{C}_\Delta(z)$.*

Proof. Let $\{x_1, \dots, x_k\} \subset \partial\mathcal{C}(z)$ denote the set of extremal points for the distance function $d_L(z, y)$.

For each $1 \leq i \leq k$, let $\Delta(z, z_1^i, \dots, z_n^i) \in \Delta^{(n)}(\mathcal{N}_X)$ denote the n -simplex defined by the center x_i , so that $\{z_1^i, \dots, z_n^i\} \subset \mathcal{V}_X^1(z)$. The claim is that the intersection $|\Delta(z, z_1^i, \dots, z_n^i)| \cap \partial\mathcal{C}(z)$ is a topological ball with “center” x_i , and the union of all these boundary regions for $1 \leq i \leq k$ is a closed covering of $\partial\mathcal{C}(z)$. Thus,

$$\mathcal{C}(z) \subset \bigcup_{i=1}^k |\Delta(z, z_1^i, \dots, z_n^i)|$$

Each vertex z_ℓ^i corresponds to a face of $\partial\mathcal{C}_X(z)$ as in (36) and an associated hyperplane, denoted by

$$\partial_\ell^i \mathcal{C}_X(z) = \mathcal{C}(z) \cap L(z, z_\ell^i)$$

The geodesic segment from z to z_ℓ^i intersects the face $\partial_\ell^i \mathcal{C}_X(z)$ in an interior point, denoted by \widehat{z}_ℓ^i . This geodesic segment is a boundary 1-simplex of each n -simplex that intersects this face. These n -simplices correspond to the extreme points for the distance function $d_L(z, y)$ restricted to $\partial_\ell^i \mathcal{C}_X(z)$, which we denote by $\{x_{i_1}, \dots, x_{i_m}\}$. Each such point x_{i_j} then corresponds to an n -simplex, which contains both points $\{z, z_\ell^i\}$ as vertices by Proposition 8.6. Thus, the face $\partial_\ell^i \mathcal{C}_X(z)$ is partitioned into closed regions corresponding to its intersection with the n -simplices determined by the extreme points x_{i_j} for $1 \leq j \leq m$. Thus, each face $\partial_\ell^i \mathcal{C}_X(z)$ is contained in the union of the realizations of the simplices satisfying $\Delta(z, z_1, \dots, z_n) \in \Delta^{(n)}(\mathcal{N}_X)$. The inclusion $\mathcal{C}(z) \subset \mathcal{C}_\Delta(z)$ follows. \square

8.3. Regular Delaunay simplicial complex. The simplicial complex $\Delta(\mathcal{N}_X)$ may have non-trivial $(n+1)$ -simplices, where some collection of $(n+1)$ -hyperplanes satisfy

$$L(z_0, z_1) \cap \dots \cap L(z_0, z_{n+1}) \cap D_{\mathcal{F}}(z_0, \lambda_{\mathcal{F}}) \neq \emptyset.$$

This is a degenerate condition, as typically every collection of $(n+1)$ -hyperplanes in $D_L(z_0, \lambda_{\mathcal{F}})$ should have empty intersection. This motivates the following definition.

DEFINITION 8.8. *The simplicial complex $\Delta(\mathcal{N}_X)$ is regular if $\Delta^{(n+1)}(\mathcal{N}_X) = \emptyset$. We say that the net \mathcal{N}_X is regular if $\Delta(\mathcal{N}_X)$ is regular.*

Much of the technical work in later sections where extensions of \mathcal{N}_X to a net $\mathcal{N}'_{X'}$ contained in a leaf L' near to L are constructed, is to give conditions on a regular net \mathcal{N}_X such that for a net $\mathcal{N}'_{X'}$ sufficiently close to \mathcal{N}_X , the Delaunay simplicial complex $\Delta(\mathcal{N}'_{X'})$ is also regular.

9. FOLIATED VORONOI STRUCTURE

In this section, we introduce transversally parametrized versions of the Voronoi cell decomposition and the corresponding Delaunay triangulation associated to a (d_1, d_2) -net $\mathcal{N}_K \subset K \subset L$. The key idea is extend the constructions above to a net in a subset of $\tilde{K} \subset \tilde{L} \subset \mathfrak{N}_0$, then introduce a parametrized set of vertices in an open neighborhood of \tilde{K} in \mathfrak{N}_0 . Such an extension gives rise to a collection of transversals to \mathcal{F} which satisfy regularity conditions with respect to the net and the construction of the Voronoi cells and the Delaunay triangulation. This culminates in the definition of a *nice stable transversal* \mathcal{X} in Definition 9.7.

We assume a leaf $L_0 \subset \mathfrak{M}$ is given, with basepoint $x_0 \in L_0$ and holonomy covering \tilde{L}_0 . In particular, recall from Section 5.1 that \mathcal{M}_0 is the given (e_1, e_2) -net for L_0 with $e_2 = \epsilon_{\mathcal{U}}^{\mathcal{F}}/4$, such that for each $z \in \mathcal{M}_0$ there is an index $1 \leq i_z \leq \nu$ so that $D_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$. Let $\tilde{\mathcal{M}}_0 = \Pi^{-1}(\mathcal{M}_0)$ be the lifted (e_1, e_2) -net for \tilde{L}_0 . The points of $\tilde{\mathcal{M}}_0$ are denoted by \tilde{z} , where \tilde{z} is a lift of $z \in \mathcal{M}_0$, and $\tilde{x}_0 \in \tilde{\mathcal{M}}_0$ is the lift of the basepoint $x_0 \in L_0$.

Let $\tilde{\mathfrak{N}}_0$ be the foliated microbundle associated to the net $\tilde{\mathcal{M}}_0$, with notations as in Section 5.1. For each $\tilde{z} \in \tilde{\mathcal{M}}_0$, we then have that $\tilde{U}_{\tilde{z}} = \overline{U}_{i_z} \times \{\tilde{z}\}$ is the corresponding foliation chart for $\tilde{\mathfrak{N}}_0$. The Riemannian metric on leaves in $\tilde{\mathfrak{N}}_0$ is induced by the local covering maps to leaves in \mathfrak{M} .

For $\tilde{z} \in \tilde{\mathcal{M}}_0$ and $\tilde{y} = (x, \tilde{z}) \in \tilde{U}_{\tilde{z}}$, let $\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{y}) = \mathcal{P}_{i_z}(x) \times \{\tilde{z}\}$ denote the plaque of $\tilde{U}_{\tilde{z}}$ containing \tilde{y} . Note that by choice, $D_{\tilde{L}_0}(\tilde{z}, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset \tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z})$ for each $\tilde{z} \in \tilde{\mathcal{M}}_0$, and as \mathcal{M}_0 is e_2 -dense, the collection $\{\tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z}) \mid \tilde{z} \in \tilde{\mathcal{N}}_0\}$ is an open covering of \tilde{L}_0 .

Let $\tilde{K} \subset \tilde{L}_0$ be a connected compact subset which is a union of plaques, such that the composition $\iota_0: \tilde{K} \subset \tilde{L}_0 \rightarrow L_0 \subset \mathfrak{M}$ is injective with image K . Assume there is given a (d_1, d_2) -net \mathcal{N}_K for K , which lifts to a (d_1, d_2) -net $\tilde{\mathcal{N}}_K$ for \tilde{K} .

We next introduce a sequence of basic concepts used in our constructions. First is the notion of transversals which are in “standard form” with respect to the chosen foliation covering of \mathfrak{M} .

DEFINITION 9.1. A closed subset $\mathcal{X} \subset \mathfrak{M}$ is a *standard transversal* if $\mathcal{X} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_p$ is a disjoint union, where for each $1 \leq \ell \leq p$, there exists a foliation chart φ_{i_ℓ} , clopen subset $X_\ell \subset \mathfrak{T}_{i_\ell}$ and basepoint $v_\ell \in (-1, 1)^n$ such that $\mathcal{X}_\ell = \varphi_{i_\ell}^{-1}(v_\ell, X_\ell)$.

DEFINITION 9.2. A closed subset $\tilde{\mathcal{X}} \subset \tilde{\mathfrak{N}}_0$ is a *standard transversal* if $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_1 \cup \dots \cup \tilde{\mathcal{X}}_p$ is a disjoint union, where for each $1 \leq \ell \leq p$, there exists $\tilde{z}_\ell \in \tilde{\mathcal{M}}_0$ so that for the foliation chart $\tilde{\varphi}_{\tilde{z}_\ell}: \tilde{U}_{\tilde{z}_\ell} \rightarrow [-1, 1]^n \times \mathfrak{T}_{\tilde{z}_\ell}$ there is a clopen subset $X_\ell \subset \mathfrak{T}_{\tilde{z}_\ell}$ and basepoint $v_\ell \in (-1, 1)^n$ such that $\tilde{\mathcal{X}}_\ell = \tilde{\varphi}_{\tilde{z}_\ell}^{-1}(v_\ell, X_\ell)$.

Note that this definition just ensures that the sets $\tilde{\mathcal{X}}_\ell$ have a standard form in local coordinates, but does not assert that the sets form a complete transversal for $\tilde{\mathfrak{N}}_0$. We consider next the standard transversals which are “holonomy invariant” in $\tilde{\mathfrak{N}}_0$.

DEFINITION 9.3. A standard transversal with $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_1 \cup \dots \cup \tilde{\mathcal{X}}_p \subset \tilde{\mathfrak{N}}_0$ is *invariant* if for each $1 < \ell \leq p$, there exists a leafwise path $\tilde{\gamma}_\ell: [0, 1] \rightarrow \tilde{L}_0$ with $\tilde{\gamma}_\ell(0) = \tilde{z}_1$ and $\tilde{\gamma}_\ell(1) = \tilde{z}_\ell$ such that $X_1 \subset D(h_{\tilde{\gamma}_\ell})$ and $X_\ell = h_{\tilde{\gamma}_\ell}(X_1)$.

Recall that the induced foliation $\tilde{\mathcal{F}}$ on $\tilde{\mathfrak{N}}_0$ is without germinal holonomy, so this notion is independent of the chosen paths $\tilde{\gamma}_\ell$, as long as the domain conditions are satisfied. The next conditions are concerned with the extensions of the notions of Sections 7 and 8. Recall that if $x, y \in \mathfrak{M}$ and are not on the same leaf, then $d_{\mathcal{F}}(x, y) = \infty$, and similarly for $\tilde{x}, \tilde{y} \in \tilde{\mathfrak{N}}_0$.

DEFINITION 9.4. Let $\tilde{\mathfrak{R}} \subset \tilde{\mathfrak{N}}_0$ be a given closed subset, and suppose that $\tilde{\mathcal{X}} \subset \tilde{\mathfrak{R}}$ is a standard transversal, defined as above. Then $\tilde{\mathcal{X}}$ is (d_1, d_2) -uniform on $\tilde{\mathfrak{R}}$ if there exists $0 < d_1 < d_2 \leq \lambda_{\mathcal{F}}/5$ such that for each $\tilde{x} \neq \tilde{y} \in \tilde{\mathcal{X}}$ we have $d_{\mathcal{F}}(\tilde{x}, \tilde{y}) \geq d_1$, and for each $\tilde{y} \in \tilde{\mathfrak{R}}$ there exists $\tilde{x} \in \tilde{\mathcal{X}}$ with $d_{\mathcal{F}}(\tilde{x}, \tilde{y}) \leq d_2$.

The (d_1, d_2) -uniform assumption above implies that $\tilde{\mathcal{X}}$ is a complete transversal for $\tilde{\mathfrak{R}}$, as every point lies within leafwise distance d_2 of a point of $\tilde{\mathcal{X}}$.

Now assume there is given a (d_1, d_2) -uniform transversal $\tilde{\mathcal{X}}$ for $\tilde{\mathfrak{R}}$. The nearest-neighbor distance function κ_L extends to a leafwise function,

$$(40) \quad \kappa_{\tilde{\mathcal{F}}}(\tilde{y}) = \inf \left\{ d_{\tilde{\mathcal{F}}}(\tilde{x}, \tilde{y}) \mid \tilde{x} \in \tilde{\mathcal{X}} \right\},$$

Then we extend the definition of the Voronoi cells to holonomy coverings by setting, for $\tilde{y} \in \tilde{\mathfrak{R}}$,

$$(41) \quad \mathcal{C}_{\tilde{\mathfrak{R}}}(\tilde{y}) = \left\{ \tilde{z} \in \tilde{\mathfrak{R}} \mid d_{\tilde{\mathcal{F}}}(\tilde{z}, \tilde{y}) = \kappa_{\tilde{\mathcal{F}}}(\tilde{z}) \right\}.$$

In other words, $\mathcal{C}_{\tilde{\mathfrak{R}}}(\tilde{x})$ is the Voronoi cell in $\tilde{L}_{\tilde{x}} \cap \tilde{\mathfrak{R}}$ defined by the net $\tilde{\mathcal{X}}(x) = \tilde{\mathcal{X}} \cap \tilde{\mathfrak{R}}$, which consists of $\tilde{y} \in \tilde{L}_{\tilde{x}} \cap \tilde{\mathfrak{R}}$ which are closer to \tilde{x} in the leafwise metric $d_{\tilde{\mathcal{F}}}$ than to any other point of $\tilde{\mathcal{X}}$. By the definition of the (d_1, d_2) -uniform transversal $\tilde{\mathcal{X}}$ for $\tilde{\mathfrak{R}}$, each $\tilde{y} \in \tilde{\mathfrak{R}}$ belongs to at least one such cell.

The leafwise Voronoi cells $\mathcal{C}_{\tilde{\mathfrak{R}}}(\tilde{x})$ can be organized into *Voronoi cylinders* using the decomposition $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_1 \cup \dots \cup \tilde{\mathcal{X}}_p$. For $1 \leq \ell \leq p$ define the *Voronoi cylinder* by

$$(42) \quad \mathfrak{C}_{\tilde{\mathfrak{R}}}^\ell = \bigcup_{\tilde{x} \in \tilde{\mathcal{X}}_\ell} \mathcal{C}_{\tilde{\mathfrak{R}}}(\tilde{x})$$

We thus obtain the *Voronoi decomposition* $\tilde{\mathfrak{R}} = \mathfrak{C}_{\tilde{\mathfrak{R}}}^1 \cup \dots \cup \mathfrak{C}_{\tilde{\mathfrak{R}}}^p$ associated to $\tilde{\mathcal{X}}$.

The notion of the *star-neighborhood* of a Voronoi cell, given in Definition 7.6, extends immediately to the Voronoi cells in each leaf. First, for $\tilde{x} \in \tilde{\mathcal{X}}$ introduce the *vertex-set*

$$(43) \quad \mathcal{V}_{\tilde{\mathfrak{R}}}(\tilde{x}) = \{ \tilde{y} \in \tilde{\mathfrak{R}} \mid \mathcal{C}_{\tilde{\mathfrak{R}}}(\tilde{y}) \cap \mathcal{C}_{\tilde{\mathfrak{R}}}(\tilde{x}) \neq \emptyset \}.$$

The star-neighborhood of the Voronoi cell $\mathcal{C}_{\tilde{\mathfrak{R}}}(\tilde{x})$ is the set $\mathcal{S}_{\tilde{\mathfrak{R}}}(\tilde{x}) = \bigcup_{\tilde{y} \in \mathcal{V}_{\tilde{\mathfrak{R}}}(\tilde{x})} \mathcal{C}_{\tilde{\mathfrak{R}}}(\tilde{y})$. Then for each

$1 \leq \ell \leq p$, define the *star-neighborhood* of the cylinder $\mathfrak{C}_{\tilde{\mathfrak{R}}}^\ell$ by

$$(44) \quad \mathfrak{S}_{\tilde{\mathfrak{R}}}^\ell = \bigcup_{\tilde{x} \in \tilde{\mathcal{X}}_\ell} \mathcal{S}_{\tilde{\mathfrak{R}}}(\tilde{x}).$$

Lemma 7.7 shows that for a point $x \in \mathcal{N}_K$, the star-neighborhood $\mathcal{S}_K(x) \subset B_L(x, 3d_2)$ so that $d_2 \leq \lambda_{\mathcal{F}}/5$ implies $\mathcal{S}_K(x)$ is contained in some coordinate chart U_z for $z \in \mathcal{M}_0$. For the star-neighborhood $\mathfrak{S}_{\tilde{\mathfrak{R}}}^\ell$ of the cylinder $\mathfrak{C}_{\tilde{\mathfrak{R}}}^\ell$, the conclusion that it is contained in some foliation chart for \mathfrak{M} is not a priori satisfied, and this condition is imposed as one of our assumptions. It will later be checked that it is satisfied for the transversals constructed.

DEFINITION 9.5. *Let $\tilde{\mathfrak{R}} \subset \tilde{\mathfrak{M}}_0$ be a given closed subset, and suppose that $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_1 \cup \dots \cup \tilde{\mathcal{X}}_p$ is a uniform standard transversal. Then we say that $\tilde{\mathcal{X}}$ is centered if for each $1 \leq \ell \leq p$ there is a coordinate chart U_{ν_ℓ} for $\nu_\ell = i_{\tilde{x}}$ for some $\tilde{x} \in \mathcal{M}_0$ such that $\mathfrak{S}_{\tilde{\mathfrak{R}}}^\ell \subset U_{\nu_\ell}$.*

A standard transversal $\tilde{\mathcal{X}}$ for $\tilde{\mathfrak{R}}$ is said to be nice if it is (d_1, d_2) -uniform, invariant and centered.

Next, define the leafwise simplicial complex $\Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$ associated to a nice transversal $\tilde{\mathcal{X}}$ for $\tilde{\mathfrak{R}}$. The method of inscribed spheres adapts immediately, as follows.

DEFINITION 9.6. *Let $\tilde{\mathcal{X}}$ be a nice transversal for $\tilde{\mathfrak{R}}$. The collection of points $\{\tilde{z}_0, \dots, \tilde{z}_k\} \subset \tilde{\mathcal{X}}$ defines a k -simplex $\Delta(\tilde{z}_0, \dots, \tilde{z}_k) \in \Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$ if there exists $\tilde{z} \in \tilde{L}_{\tilde{z}_0}$ and $r \leq d_2$ such that*

$$(45) \quad \{\tilde{z}_0, \dots, \tilde{z}_k\} \subset S_{\tilde{\mathcal{F}}}(\tilde{z}, r) \cap \tilde{\mathcal{X}} \quad , \quad B_{\tilde{\mathcal{F}}}(\tilde{z}, r) \cap \tilde{\mathcal{X}} = \emptyset$$

Again, note that $d_{\tilde{\mathcal{F}}}(\tilde{x}, \tilde{y}) = \infty$ if \tilde{x} and \tilde{y} lie on distinct leaves, so $\Delta(\tilde{z}_0, \dots, \tilde{z}_k) \in \Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$ implies that $\{\tilde{z}_0, \dots, \tilde{z}_k\} \subset \tilde{L}_{\tilde{z}_0}$. Consequently, $\Delta(\tilde{z}_0, \dots, \tilde{z}_k)$ can also be considered as a k -simplex for $\tilde{\mathcal{X}} \cap \tilde{L}_{\tilde{z}_0}$, so that $\Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$ consists of a union of simplices contained in the leaves of $\tilde{\mathcal{F}}$. The key

question is then, given $\Delta(\tilde{z}_0, \dots, \tilde{z}_k) \in \Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$, is it contained in a transverse family of simplices? We make this property precise, as it is fundamental.

Let $1 \leq j_0, \dots, j_k \leq p$ be indices such that $\tilde{z}_\ell \in \tilde{\mathcal{X}}_{j_\ell}$ for $0 \leq \ell \leq k$. In particular, $\tilde{z}_0 \in \tilde{\mathcal{X}}_{j_0} \subset U_{i_{j_0}}$ and thus $\tilde{z}_0 \in \tilde{\mathcal{P}}_{i_{j_0}}(\tilde{z}_0)$. As $\tilde{\mathcal{X}}$ is centered, we can assume $z_\ell \in \mathcal{P}_{i_{j_0}}(z_0)$ for $1 \leq \ell \leq k$ and $\mathcal{X}_{j_\ell} \subset U_{i_{j_0}}$.

Note that for $\ell \neq \ell'$ the sets \mathcal{X}_{j_ℓ} are $\mathcal{X}_{j_{\ell'}}$ are disjoint by the (d_1, d_2) -net hypothesis.

Let $z'_0 \in \mathcal{X}_{j_0}$. Let $\mathcal{P}_{i_{j_0}}(z'_0)$ denote the plaque of $U_{i_{j_0}}$ containing z'_0 . For each $1 \leq \ell \leq n$, let $z'_\ell = \mathcal{X}_{i_\ell} \cap \mathcal{P}_{i_{j_0}}(z'_0)$ be the unique point of \mathcal{X}_{i_ℓ} contained in the plaque defined by z'_0 . Observe that the points z'_ℓ depend continuously on $z'_0 \in \mathcal{X}_{j_0}$.

DEFINITION 9.7. *Let $\tilde{\mathcal{X}}$ be a nice transversal for $\tilde{\mathfrak{R}}$. Then $\tilde{\mathcal{X}}$ is stable if for each k -simplex $\Delta(\tilde{z}_0, \dots, \tilde{z}_k) \in \Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$ and $\tilde{z}'_0 \in \mathcal{X}_{j_0}$, we have $\Delta(\tilde{z}'_0, \dots, \tilde{z}'_k) \in \Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$.*

At first inspection, stability of simplices for a Delaunay triangulation associated to a $\tilde{\mathcal{X}}$ seems to be intuitively clear, and in fact this is basically correct for dimension $n \leq 2$. The difficulty is that for $n > 2$, as the transverse coordinate $\tilde{z}'_0 \in \mathcal{X}_{j_0}$ varies, “small variations” of the spacings of the net points of $\tilde{\mathcal{X}} \cap \tilde{L}_{\tilde{z}'_0}$ may result in an abrupt change in the Delaunay simplicial structure, if some face of a Voronoi cell has too small of a diameter relative to the size of the variation. Consequently, the existence of a nice stable transversal for $n > 2$ requires delicate estimates in its construction.

10. CONSTRUCTIONS OF TRANSVERSE CANTOR FOLIATIONS

In this section, we show how the existence of a nice stable transversal is used to construct a transverse Cantor foliation \mathcal{H} for a subset $\mathfrak{B} \subset \tilde{\mathfrak{N}}_0$ as given in (50) below. This is a key step in the proofs of Theorems 1.1 and 1.3, which are given in Section 19.

Let $x \in \mathfrak{M}$ and $L_x \subset \mathfrak{M}$ be the leaf through x , with holonomy covering $\Pi: \tilde{L}_x \rightarrow L_x$.

For $x \in \mathfrak{M}$, assume there is given a connected compact subset $K_x \subset L_x$ such that there is $\tilde{K}_x \subset \tilde{L}_x$ such that $\iota_x: \tilde{K}_x \subset \tilde{L}_x \rightarrow L_x \subset \mathfrak{M}$ is injective with image K_x . In order to construct a transverse Cantor foliation for \tilde{K}_x , we introduce a sequence of modifications of the set \tilde{K}_x , first to expand the set, then translate it to a leaf L_0 without holonomy, resulting in a set $\tilde{K}_0 \subset \tilde{L}_0$ for which we assume there is a nice stable transversal $\tilde{\mathcal{X}}$, from which the construction proceeds.

Without loss of generality, we may assume that $x \in K_x$, and then let $\tilde{x} \in \tilde{K}_x$ be the lift of x . Let U_{i_x} be a foliation chart as in (2.3) such that $B_{\mathfrak{M}}(x, \epsilon_U) \subset U_{i_x}$, and set $w_x = \pi_{i_x}(x) \in \mathfrak{T}_{i_x}$. Assume that the transverse models for \mathcal{F} are Cantor sets, and thus w_x is not an isolated point.

The germinal holonomy of the leaf L_x is given by the isotropy subgroup $\Gamma_{\mathcal{F}}^{w_x}$ in (14), which is represented by the elements of the pseudogroup $\mathcal{G}_{\mathcal{F}}^*$ which fix w_x . Then for every clopen neighborhood $w_x \in V_x \subset \mathfrak{T}_{i_x}$, Theorem 3.6 implies there exists $w_0 \in V_x$ such that the leaf corresponding to w_0 is without holonomy. If L_x is without holonomy, then we may take $w_0 = w_x$.

Given complete separable metric space (X, d_X) , a proper subset $Y \subset X$ and $\epsilon > 0$, introduce the notion of the ϵ -*penumbra* of Y in X ,

$$(46) \quad \text{Pen}_X(Y, \epsilon) = \{x \in X \mid d_X(x, Y) \leq \epsilon\}.$$

That is, $\text{Pen}_X(Y, \epsilon)$ is the closed subset of X consisting of all points within distance ϵ of Y . We apply this construction to $Y = \tilde{K}_x \subset X = \tilde{L}_x$ for $\epsilon = \lambda_{\mathcal{F}}$. Then by definition, for every point $\tilde{y} \in \tilde{K}_x$ there is a “large enough” neighborhood $D_{\mathcal{F}}(\tilde{y}, \lambda_{\mathcal{F}}) \subset \text{Pen}_{\mathcal{F}}(\tilde{K}_x, \lambda_{\mathcal{F}})$. Finally, let \hat{K}_x be the plaque saturation of $\text{Pen}_{\mathcal{F}}(\tilde{K}_x, \lambda_{\mathcal{F}})$ in \tilde{L}_x , so

$$(47) \quad \hat{K}_x = \bigcup \left\{ \tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z}) \mid \tilde{z} \in \tilde{L}_x, \tilde{\mathcal{P}}_{\tilde{z}}(\tilde{z}) \cap \text{Pen}_{\mathcal{F}}(\tilde{K}_x, \lambda_{\mathcal{F}}) \neq \emptyset \right\}$$

Let \widehat{R}_K denote the diameter of the set \widehat{K}_x in \widetilde{L}_x . It follows that $\widehat{K}_x \subset D_{\widetilde{\mathcal{F}}}(\widetilde{x}, \widehat{R}_K)$. Then recall from Proposition 4.9 that given $\epsilon > 0$, there exists $0 < \delta(\epsilon, \widehat{R}_K) \leq \epsilon$ so that for any clopen neighborhood V_x with diameter at most $\delta(\epsilon, \widehat{R}_K)$ in \mathfrak{T}_{i_x} , the set V_x is \widehat{K}_x -admissible, as defined by Definition 5.8.

The choice of $\epsilon > 0$ and the clopen neighborhood V_x will be specified in later sections, based on the radius \widehat{R}_K and estimates derived from the leafwise Riemannian geometry. For now, we assume they are given. Then choose $w_0 \in V_x$ to be a point whose leaf $L_0 \subset \mathfrak{M}$ is without holonomy. Let $\Pi: \widetilde{L}_0 \rightarrow L_0$ be the holonomy cover, which is a diffeomorphism.

Let $\widetilde{\mathcal{M}}_0$ be an (e_1, e_2) -net for \widetilde{L}_0 as in Section 5.1, where $e_2 = \epsilon_{\mathcal{U}}^{\mathcal{F}}/4$, and let \mathcal{M}_0 be the projection of the net to L_0 . Select the collection of coordinate charts $\{U_{i_z} \mid z \in \mathcal{M}_0\}$ for \mathfrak{M} as in the proof of Lemma 5.2. Then for each $\widetilde{z} \in \widetilde{\mathcal{M}}_0$, there is an index $1 \leq i_z \leq \nu$ such that for $\widetilde{U}_{\widetilde{z}} = \overline{U}_{i_z} \times \{\widetilde{z}\}$, as discussed in Section 5.2, we have $B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$ where $z = \Pi(\widetilde{z})$. Form the Reeb neighborhood $\widetilde{\mathfrak{N}}(\widehat{K}_x, V_x) \subset \widetilde{\mathfrak{N}}_0$ as in Definition 5.8.

Let \widehat{K}_0 be the connected compact subset of the holonomy cover \widetilde{L}_0 obtained by taking the union of the plaques in \widetilde{L}_0 which are contained in $\widetilde{\mathfrak{N}}(\widehat{K}_x, V_x)$, so \widehat{K}_0 is a “translation” of \widehat{K}_x to the leaf \widetilde{L}_0 .

THEOREM 10.1. *For $0 < d_1 < d_2 \leq \lambda_{\mathcal{F}}/5$, assume there is given a nice stable (d_1, d_2) -uniform transversal $\widetilde{\mathcal{X}}$ for $\mathfrak{N}(\widehat{K}_x, V_x)$. Then there exists a foliated homeomorphism into,*

$$(48) \quad \Phi: \widehat{K}_x \times V_x \rightarrow \widetilde{\mathfrak{N}}(\widehat{K}_x, V_x)$$

such that the images $\Phi(\{\widetilde{y}\} \times V_x)$, for $\widetilde{y} \in \widetilde{K}_x$ define a continuous family of Cantor transversals for $\widetilde{\mathcal{F}}|_{\widetilde{\mathfrak{N}}(\widehat{K}_x, V_x)}$ which extend the transversals in $\widetilde{\mathcal{X}}$.

Thus, the assumption there is a nice stable transversal for $\widehat{K}_0 \subset \mathfrak{N}(\widehat{K}_x, V_x)$ implies there is a transverse Cantor foliation $\widetilde{\mathcal{H}}$ defined on some open neighborhood of \widehat{K}_x in $\mathfrak{N}(\widehat{K}_x, V_x)$. The proof of Theorem 10.1 occupies the rest of this section. Then in subsequent sections, we establish criteria for when the hypotheses of Theorem 10.1 are satisfied.

Assume there is given a nice stable (d_1, d_2) -uniform transversal $\widetilde{\mathcal{X}} = \widetilde{\mathcal{X}}_1 \cup \dots \cup \widetilde{\mathcal{X}}_p$ for $\widehat{K}_0 \subset \mathfrak{N}(\widehat{K}_x, V_x)$. We continue with the notations of Sections 8 and 9.

Let $\Delta(\widetilde{z}_0, \dots, \widetilde{z}_k) \in \Delta_{\mathcal{F}}(\widetilde{\mathcal{X}})$ be given, with $\widetilde{z}_\ell \in \widetilde{\mathcal{X}}_{i_\ell}$. Then $\{\widetilde{z}_0, \dots, \widetilde{z}_k\} \subset \widetilde{U}_{i_0}$ so we have $\ell \neq \ell'$ implies that $i_\ell \neq i_{\ell'}$. Without loss of generality, we may re-index the vertices so that $\ell < \ell'$ implies $i_\ell < i_{\ell'}$. That is, the indexing of the sets $\widetilde{\mathcal{X}}_\ell$ yields an ordering of the vertices of $\Delta(\widetilde{z}_0, \dots, \widetilde{z}_k)$. This is the “local ordering” referred to in Remark 8.5.

The transversal $\widetilde{\mathcal{X}}$ defines the leaves of the Cantor foliation $\widetilde{\mathcal{H}}$ through each vertex $\{\widetilde{z}_0, \dots, \widetilde{z}_k\}$. We next show how to extend this finite collection of leaves to a foliation through the faces and interior of the simplex $\Delta(\widetilde{z}_0, \dots, \widetilde{z}_k)$.

For each $\widetilde{z}'_0 \in \widetilde{\mathcal{X}}_{i_0}$ and $1 \leq \ell \leq k$, let $\widetilde{z}'_\ell = \widetilde{\mathcal{P}}_{i_0}(\widetilde{z}'_0) \cap \widetilde{\mathcal{X}}_{i_\ell}$. The stable hypothesis then implies that $\Delta(\widetilde{z}'_0, \dots, \widetilde{z}'_k) \in \Delta_{\mathcal{F}}(\widetilde{\mathcal{X}})$.

By Lemma 8.2, for each $\widetilde{z}'_0 \in \widetilde{\mathcal{X}}_{i_0}$ there exists a geodesic filling map $\sigma_{k, \widetilde{z}'_0}: \Delta^k \rightarrow \widetilde{\mathcal{P}}_{i_0}(\widetilde{z}'_0) \subset \widetilde{L}_{\widetilde{z}'_0}$ associated to $\Delta(\widetilde{z}'_0, \dots, \widetilde{z}'_k)$ which is natural with respect to the face maps. It follows that the map

$$(49) \quad \Sigma_{i_0}: \Delta^k \times \widetilde{\mathcal{X}}_{i_0} \rightarrow \mathfrak{R}: (\vec{v}, \widetilde{z}'_0) \mapsto \sigma_{k, \widetilde{z}'_0}(\vec{v}) \quad , \quad \widetilde{z}'_0 \in \widetilde{\mathcal{X}}_{i_0} \quad , \quad \vec{v} \in \Delta^k$$

is continuous. Then for each $\vec{v} \in \Delta^k$ and $\widetilde{z}'_0, \widetilde{z}''_0 \in \widetilde{\mathcal{X}}_{i_0}$ define $\sigma_{k, \widetilde{z}'_0}(\vec{v}) \approx \sigma_{k, \widetilde{z}''_0}(\vec{v})$. The equivalence class of $\sigma_{k, \widetilde{z}'_0}(\vec{v})$ defines a Cantor transversal through the point, which is a leaf of $\widetilde{\mathcal{H}}$.

The foliation $\widetilde{\mathcal{H}}$ is defined by the equivalence classes of points in the interiors of the geometric realizations of the simplices in $\Delta_{\mathcal{F}}(\widetilde{\mathcal{X}})$. On the faces of adjacent simplices, the local orderings are compatible, so the geodesic filling maps agree, and thus so does the equivalence relation \approx .

We underline some points of this construction. First, for each $1 \leq \ell \leq p$ and $\widetilde{z}, \widetilde{z}' \in \widetilde{\mathcal{X}}_\ell$ then $\widetilde{z} \approx \widetilde{z}'$. That is, each transversal $\widetilde{\mathcal{X}}_\ell$ is a leaf of the foliation $\widetilde{\mathcal{H}}$.

Second, for each 1-simplex $\Delta(\tilde{z}_0, \tilde{z}_1) \in \Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$ the equivalence relation \approx identifies points with the same barycentric coordinate on the unique geodesic ray joining \tilde{z}'_1 to \tilde{z}'_0 where $\Delta(\tilde{z}'_0, \tilde{z}'_1)$ is the transverse transport of the given 1-simplex. Thus, \approx is independent of the ordering when restricted to the 1-skeleton of the leafwise triangulation $\Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$. If \mathcal{F} is an orientable foliation by 1-dimensional leaves, that is, it is defined by a flow, then we are done, and the equivalence relation \approx depends canonically on the choice of the uniform transversal $\tilde{\mathcal{X}}$, but is independent of its ordering.

If the leaves of \mathcal{F} have dimension $n > 1$, then the “spanning geodesic procedure” in the proof of Lemma 8.2 may well depend upon the ordering of the vertices in each simplex. However, the “local ordering” of the vertices in simplices is determined by the choice of the transversal $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_1 \cup \dots \cup \tilde{\mathcal{X}}_p$. Thus \approx is well-defined, assuming the choice of the transversal $\tilde{\mathcal{X}}$ with its ordering.

It remains to make precise the set of points in $\mathfrak{N}(\hat{K}_x, V_x)$ which are in the domain of the equivalence relation \approx , and show that \tilde{K}_x is contained in this domain.

For each $\tilde{y} \in \tilde{\mathcal{X}}$, with leaf $\tilde{L}_{\tilde{y}}$ containing it, the intersection $\tilde{\mathcal{N}}_{\tilde{y}} = \tilde{\mathcal{X}} \cap \tilde{L}_{\tilde{y}}$ is a (d_1, d_2) -net for $\mathfrak{N}(\hat{K}_x, V_x) \cap \tilde{L}_{\tilde{y}}$ by Definition 9.4. Define the function $\kappa_{\tilde{\mathcal{F}}}$ as in (40) and the Voronoi cell, as in (41),

$$\mathcal{C}(\tilde{y}) = \{\tilde{z} \in \tilde{L}_{\tilde{y}} \mid d_{\mathcal{F}}(\tilde{z}, \tilde{y}) = \kappa_{\tilde{\mathcal{F}}}(\tilde{z})\}$$

Let $\tilde{\mathcal{N}}_{\tilde{y}}^* \subset \tilde{\mathcal{N}}_{\tilde{y}}$ be the subset of points for which $\mathcal{C}(\tilde{y}) \subset \mathfrak{N}(\hat{K}_x, V_x)$, and for the special case of L_0 set $\tilde{\mathcal{N}}_0 = \tilde{\mathcal{X}} \cap \tilde{L}_0$ with $\tilde{\mathcal{N}}_0^* \subset \tilde{\mathcal{N}}_0$.

Let $\mathcal{C}_{\Delta}(\tilde{y})$ be the simplicial cone of $\tilde{y} \in \tilde{\mathcal{X}}$ in the complex $\Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$ as defined leafwise by (39). Then the stability assumption on $\tilde{\mathcal{X}}$ implies the simplicial complex $\Delta(\tilde{\mathcal{N}}_{\tilde{y}})$ is stable, and we define

$$(50) \quad \mathfrak{B} = \bigcup_{\tilde{z}_i \in \tilde{\mathcal{N}}_0^*} \bigcup_{\tilde{y} \in \tilde{\mathcal{X}}_i} \mathcal{C}_{\Delta}(\tilde{y})$$

Then the equivalence relation \approx is defined on \mathfrak{B} by definition, so \mathfrak{B} admits a Cantor foliation \mathcal{H} .

The proof of Theorem 10.1 then follows from

LEMMA 10.2. *Assume that $\tilde{\mathcal{X}}$ is given a nice stable (d_1, d_2) -uniform transversal $\tilde{\mathcal{X}}$ for $\mathfrak{N}(\hat{K}_x, V_x)$. Then $\tilde{K}_x \subset \mathfrak{B}$.*

Proof. By Definition 9.4 and the construction of $\mathfrak{N}(\hat{K}_x, V_x)$, for each $\tilde{y} \in \tilde{K}_x$ there exists $\tilde{z} \in \tilde{\mathcal{X}}_{\ell}$ such that $d_{\tilde{\mathcal{F}}}(\tilde{y}, \tilde{z}) \leq d_2$. We may assume in addition that \tilde{z} is a closest point in $\tilde{\mathcal{X}}$, so that $\tilde{y} \in \mathcal{C}(\tilde{z})$. Then note that

$$D_{\tilde{\mathcal{F}}}(\tilde{y}, \lambda_{\mathcal{F}}) \subset \text{Pen}_{\mathcal{F}}(\tilde{K}_x, \lambda_{\mathcal{F}}) \subset \hat{K}_x$$

by construction, and $d_2 \leq \lambda_{\mathcal{F}}/5$ implies that $D_{\tilde{\mathcal{F}}}(\tilde{z}, 4d_2) \subset D_{\tilde{\mathcal{F}}}(\tilde{z}, 4\lambda_{\mathcal{F}}/5) \subset \hat{K}_x$ as well.

In particular, $D_{\tilde{\mathcal{F}}}(\tilde{z}, 4d_2) \subset \hat{K}_x$, so for each $\tilde{y}' \in \mathcal{V}_{\mathfrak{N}}(\tilde{z})$ as defined in (43), we have $D_{\tilde{\mathcal{F}}}(\tilde{y}', 2d_2) \subset \hat{K}_x$. This implies $\tilde{y}' \in \tilde{\mathcal{N}}_{\tilde{y}}^*$ by the extension of Lemma 7.5. Consequently, the star-neighborhood, as defined in (44) for $\mathfrak{N} = \mathfrak{N}(\hat{K}_x, V_x)$, satisfies $\mathfrak{S}_{\mathfrak{N}}^{\ell}(\tilde{z}) \subset B_{\tilde{L}_x}(\tilde{z}, 3d_2) \subset \hat{K}_x$.

By Proposition 8.7, for $\tilde{z} \in \tilde{\mathcal{N}}_0^*$ we have $\mathcal{C}(\tilde{z}) \subset \mathcal{C}_{\Delta}(\tilde{z})$, where $\mathcal{C}_{\Delta}(\tilde{z})$ is the simplicial cone of \tilde{z} in the complex $\Delta_{\mathcal{F}}(\tilde{\mathcal{X}})$ as defined by (39). Thus we have

$$(51) \quad \tilde{y} \in \mathcal{C}(\tilde{z}) \subset \mathcal{C}_{\Delta}(\tilde{z}) \subset \mathfrak{B}$$

which completes the proof of Lemma 10.2 and so also Theorem 10.1. \square

It remains to show the existence of a nice stable transversals $\tilde{\mathcal{X}}$. Our approach is to develop effective estimates on the process of defining the Voronoi tessellation for a net and its associated Delaunay triangulation, and will be given in Parts III and IV that follow.

PART III - EUCLIDEAN STRUCTURES

In Part III, we consider the special case of \mathbb{R}^n with the standard Euclidean metric $d_{\mathbb{R}^n}$ and associated norm $\|\cdot\|$. The definition of the Delaunay simplicial complex in Definition 8.1 can then be expressed in terms of linear equations and estimates. We establish several key technical estimates on the center and radius of an inscribed sphere, in terms of the defining points in a net, which are fundamental for estimating stability of the Delaunay complex.

11. DELAUNAY SIMPLICES IN EUCLIDEAN GEOMETRY

Given a collection of vectors $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$ which admit an inscribed sphere with center $\omega(\vec{y}_0, \dots, \vec{y}_n)$ and radius $r(\vec{y}_0, \dots, \vec{y}_n)$, we determine conditions for which a small displacement $\{\vec{z}_0, \dots, \vec{z}_n\}$ of $\{\vec{y}_0, \dots, \vec{y}_n\}$ still uniquely defines an inscribed sphere.

11.1. Preliminaries. We consider each $\vec{x} \in \mathbb{R}^n$ as a column vector, and let $\vec{x} \bullet \vec{y} = \vec{x}^t \cdot \vec{y}$ denote the “dot-product” of two vectors, where \vec{x}^t denotes the matrix transpose of \vec{y} , and $\vec{x}^t \cdot \vec{y}$ denotes the standard matrix product.

Given a collection of n vectors, $\{\vec{a}_1, \dots, \vec{a}_n\} \subset \mathbb{R}^n$, let \mathbf{A} denote the $n \times n$ matrix with these vectors as *rows*. Denote the operator norm for \mathbf{A} by

$$\|\mathbf{A}\| = \max \{ \|\mathbf{A} \cdot \vec{x}\| \mid \vec{x} \in \mathbb{R}^n, \|\vec{x}\| = 1 \}.$$

If \mathbf{A} is a diagonal matrix with entries $\{\lambda_1, \dots, \lambda_n\}$, then the norm is calculated by

$$(52) \quad \|\mathbf{A}\| = \max \{ |\lambda_1|, \dots, |\lambda_n| \}$$

and in general, the Cauchy-Schwartz inequality yields the estimate

$$(53) \quad \|\mathbf{A}\|^2 \leq \|\vec{a}_1\|^2 + \dots + \|\vec{a}_n\|^2.$$

We recall an elementary result of linear algebra:

LEMMA 11.1. *Let $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$ and form their convex hull*

$$\Delta(\vec{y}_0, \dots, \vec{y}_n) = \{ t_0 \vec{y}_0 + \dots + t_n \vec{y}_n \mid t_0 + \dots + t_n = 1, t_i \geq 0 \}$$

Fix $0 \leq \ell \leq n$, and let $D_\ell(\vec{y}_0, \dots, \vec{y}_n)$ be the $n \times n$ -matrix whose rows are the transposes of the vectors $\vec{y}_i - \vec{y}_\ell$ for $i \neq \ell$. Then

$$(54) \quad |\det D_\ell(\vec{y}_0, \dots, \vec{y}_n)| = n! \cdot \text{Vol}(\Delta(\vec{y}_0, \dots, \vec{y}_n))$$

Proof. The volume of $\Delta(\vec{y}_0, \dots, \vec{y}_n)$ is unchanged by translation of the vertex \vec{y}_ℓ to the origin, and the result then follows by the multilinear and alternating properties of the determinant function on n -tuples of vectors in \mathbb{R}^n . \square

Now assume we are given a collection $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$ which admit an inscribed sphere with center $\omega(\vec{y}_0, \dots, \vec{y}_n)$ and radius $r(\vec{y}_0, \dots, \vec{y}_n)$. To derive the equations for the center and radius, we pick one of the vectors \vec{y}_ℓ as a “base point”. Clearly, the solutions do not depend on which vertex is chosen. It is most convenient when constructing leafwise nets in later sections to use the last vertex, so let $\ell = n$. Then for each $1 \leq k \leq n$, set $\vec{u}_k = (\vec{y}_{k-1} - \vec{y}_n)$. Then $\|\vec{u}_k\| \leq 2r(\vec{y}_0, \dots, \vec{y}_n)$ as all vectors \vec{y}_i are contained in a set with diameter $2r(\vec{y}_0, \dots, \vec{y}_n)$. Let \mathbf{U} denote the $n \times n$ matrix whose rows are the transposes of the vectors \vec{u}_k . Let $|\mathbf{U}| = |\det \mathbf{U}|$ denote the absolute value of the determinant of \mathbf{U} . Then $|\mathbf{U}| = n! \cdot \text{Vol}(\Delta(\vec{y}_0, \dots, \vec{y}_n))$ by Lemma 11.1. In particular, as will later utilize, \mathbf{U} can be calculated using vectors based at any of the vectors \vec{y}_ℓ .

Suppose there exists constants $0 < e_1 < e_2$ and $\varepsilon, \delta > 0$ such that

- (1) $e_1 \leq \|\vec{y}_i - \vec{y}_j\|$ for all $0 \leq i \neq j \leq n$,
- (2) $e_1/2 \leq r(\vec{y}_0, \dots, \vec{y}_n) \leq e_2$ and hence $\|\vec{y}_i - \vec{y}_j\| \leq 2e_2$,
- (3) $|\mathbf{U}| \geq \delta$.

Our goal is to obtain effective estimates for the inscribed sphere for a small perturbation of a given net, so assume we are given vectors $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$ such that

$$(4) \quad \|\vec{y}_i - \vec{z}_i\| < \varepsilon \text{ for all } 0 \leq i \leq n.$$

We determine values of the constants $\varepsilon, \delta > 0$ such that the points $\{\vec{z}_0, \dots, \vec{z}_n\}$ admit a unique inscribed sphere with center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ and radius $r(\vec{z}_0, \dots, \vec{z}_n)$, and obtain estimates for

$$\|\omega(\vec{z}_0, \dots, \vec{z}_n) - \omega(\vec{y}_0, \dots, \vec{y}_n)\| \quad \text{and} \quad |r(\vec{z}_0, \dots, \vec{z}_n) - r(\vec{y}_0, \dots, \vec{y}_n)|.$$

11.2. Centers and radii of inscribed spheres. The hyperplanes $L(\vec{y}_{k-1}, \vec{y}_n)$, for $1 \leq k \leq n$, are described by the equations

$$\begin{aligned} L(\vec{y}_{k-1}, \vec{y}_n) &= \{\vec{x} \in \mathbb{R}^n \mid (\vec{y}_{k-1} - \vec{y}_n) \bullet (\vec{x} - (\vec{y}_{k-1} + \vec{y}_n)/2) = 0\} \\ &= \{\vec{x} \in \mathbb{R}^n \mid \vec{u}_k \bullet \vec{x} = 1/2 \cdot \vec{u}_k \bullet \vec{u}_k + \vec{u}_k \bullet \vec{y}_n\} \\ (55) \quad &= \left\{ \xi + \vec{y}_n \mid \vec{u}_k \bullet \vec{\xi} = 1/2 \cdot \|\vec{u}_k\|^2 \right\}, \end{aligned}$$

where $\xi = \vec{x} - \vec{y}_n$ represents the coordinates for $L(\vec{y}_k, \vec{y}_n)$ with \vec{y}_n translated to the origin. The center $\vec{\omega}(\vec{y}_0, \dots, \vec{y}_n) \in \mathbb{R}^n$ is thus the solution of the system of equations

$$(56) \quad \mathbf{U} \cdot \xi = \frac{1}{2} \cdot \vec{\lambda}(\mathbf{U}) \quad \text{so} \quad \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n) = \frac{1}{2} \cdot \left\{ \mathbf{U}^{-1} \cdot \vec{\lambda}(\mathbf{U}) \right\} + \vec{y}_n,$$

where $\vec{\lambda}(\mathbf{U}) = (\|\vec{u}_1\|^2, \dots, \|\vec{u}_n\|^2)^t$ is the column vector with entries $\|\vec{u}_k\|^2$.

Similarly, let \mathbf{V} denote the $n \times n$ matrix whose rows are the transposes of the vectors $\vec{v}_k = \vec{z}_{k-1} - \vec{z}_n$ for $1 \leq k \leq n$, and set $\vec{\lambda}(\mathbf{V}) = (\|\vec{v}_1\|^2, \dots, \|\vec{v}_n\|^2)^t$. Assuming that \mathbf{V}^{-1} exists, then for $\zeta = \vec{x} - \vec{z}_n$, the solution of the matrix equation

$$(57) \quad \mathbf{V} \cdot \zeta = \frac{1}{2} \cdot \vec{\lambda}(\mathbf{V}) \quad , \quad \vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) = \frac{1}{2} \cdot \left\{ \mathbf{V}^{-1} \cdot \vec{\lambda}(\mathbf{V}) \right\} + \vec{z}_n$$

is the center for a unique inscribed sphere containing the points $\{\vec{z}_0, \dots, \vec{z}_n\}$. Our first goal is to determine when \mathbf{V} is invertible.

11.3. Existence of the inverses. We obtain a condition under which \mathbf{V} is invertible, and estimate the matrix norms $\|\mathbf{U}\|^{-1}$ and $\|\mathbf{V}^{-1}\|$.

Let $\mathbf{W} = \mathbf{V} - \mathbf{U}$ so $\mathbf{V} = \mathbf{U} + \mathbf{W}$, and set $\mathbf{Q} = \mathbf{W}\mathbf{U}^{-1}$.

LEMMA 11.2. *Assume that $\|\mathbf{Q}\| \leq 1/2$, then \mathbf{V}^{-1} exists, and $\|\mathbf{V}^{-1}\| \leq 2\|\mathbf{U}^{-1}\|$.*

Proof. Since $\mathbf{V} = (\mathbf{I} + \mathbf{Q})\mathbf{U}$, and we assume that \mathbf{U}^{-1} exists and $\|\mathbf{Q}\| < 1$, its inverse is given by

$$(58) \quad \mathbf{V}^{-1} = \mathbf{U}^{-1}(\mathbf{I} + \mathbf{Q})^{-1} = \mathbf{U}^{-1}(\mathbf{I} - \mathbf{Q} + \mathbf{Q}^2 - \mathbf{Q}^3 \pm \dots).$$

Hence, estimating the norm of the infinite sum using the triangle inequality inductively, we obtain

$$(59) \quad \|\mathbf{V}^{-1}\| \leq \|\mathbf{U}^{-1}\| \cdot \|(I - \mathbf{Q} + \mathbf{Q}^2 - \mathbf{Q}^3 \pm \dots)\| \leq \|\mathbf{U}^{-1}\|/(1 - \|\mathbf{Q}\|).$$

As $1/(1 - \|\mathbf{Q}\|) \leq 2$ this completes the proof. \square

Next, the triangle inequality and our given data yield the following estimates, where $e_3 = e_2 + \varepsilon$,

$$(60) \quad \|\vec{v}_k - \vec{u}_k\| \leq \|\vec{z}_{k-1} - \vec{y}_{k-1}\| + \|\vec{z}_n - \vec{y}_n\| \leq 2\varepsilon,$$

$$(61) \quad e_1 - 2\varepsilon \leq \|\vec{u}_k\| - \|\vec{v}_k - \vec{u}_k\| \leq \|\vec{v}_k\| \leq \|\vec{u}_k\| + \|\vec{v}_k - \vec{u}_k\| \leq 2e_2 + 2\varepsilon = 2e_3,$$

then by assumption (2) and (61)

$$(62) \quad \left| \|\vec{v}_k\|^2 - \|\vec{u}_k\|^2 \right| = |(\vec{v}_k - \vec{u}_k) \bullet (\vec{v}_k + \vec{u}_k)| \leq \|\vec{v}_k - \vec{u}_k\| \cdot (\|\vec{v}_k\| + \|\vec{u}_k\|) \leq 4\varepsilon(e_2 + e_3).$$

We then have the matrix norm estimate, that is, (53) and (61) imply

$$(63) \quad \|\mathbf{W}\| = \|\mathbf{V} - \mathbf{U}\| \leq \sqrt{\|\vec{v}_1 - \vec{u}_1\|^2 + \dots + \|\vec{v}_n - \vec{u}_n\|^2} \leq 2\varepsilon\sqrt{n}.$$

We next estimate the norm $\|\mathbf{U}^{-1}\|$. Our colleague Shmuel Friedland suggested the use of the *Hadamard determinantal inequality* in the proof of the following general estimate:

LEMMA 11.3. *Let \mathbf{A} be an $n \times n$ -matrix whose determinant has absolute value $|\mathbf{A}| > 0$, and such that each column of \mathbf{A} has norm at most C . Then*

$$(64) \quad \|\mathbf{A}^{-1}\| \leq n \cdot C^{n-1}/|\mathbf{A}|.$$

Proof. For an invertible $n \times n$ -matrix \mathbf{C} , let $0 < |\sigma_n(\mathbf{C})| \leq \dots \leq |\sigma_1(\mathbf{C})|$ denote the singular values of \mathbf{C} , ordered by their norms. Recall that $\|\mathbf{C}\|^2 = \|\mathbf{C}^t \cdot \mathbf{C}\| = |\sigma_1(\mathbf{C})|^2$.

Let $\mathbf{adj}(\mathbf{A})$ denote the adjoint of \mathbf{A} . Since $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{adj}(\mathbf{A})$ it follows that the singular values of $\mathbf{adj}(\mathbf{A})$ are all the $(n-1)$ products of the n singular values of \mathbf{A} . Hence the largest singular value for $\mathbf{adj}(\mathbf{A})$ is

$$\sigma_1(\mathbf{adj}(\mathbf{A})) = \sigma_1(\mathbf{A}) \cdots \sigma_{n-1}(\mathbf{A}).$$

Each entry of $\mathbf{adj}(\mathbf{A})$ is an $(n-1)$ minor of \mathbf{A} , and thus its absolute value is less or equal to C^{n-1} by the Hadamard determinantal inequality.

Now if $\mathbf{B} = [b_{ij}] \in \mathbb{R}^{n \times n}$ is such that the absolute value of each entry is bounded above by $\alpha > 0$, then $\|\mathbf{B}\| \leq n\alpha$, since each L^2 -norm of the column of \mathbf{B} is bounded by $\alpha\sqrt{n}$ and we apply (53).

Thus $|\sigma_1(\mathbf{adj}(\mathbf{A}))| = |\sigma_1(\mathbf{A}) \cdots \sigma_{n-1}(\mathbf{A})| \leq n \cdot C^{n-1}$, and the claim (64) follows. \square

COROLLARY 11.4. *Let $\{\vec{u}_1, \dots, \vec{u}_n\} \subset \mathbb{R}^n$ satisfy $\|\vec{u}_k\| \leq 2e_2$ for $1 \leq k \leq n$, and $|\mathbf{U}| \geq \delta$. Then*

$$(65) \quad \|\mathbf{U}^{-1}\| \leq n(2e_2)^{n-1}/|\mathbf{U}| \leq n \cdot (2e_2)^{n-1}/\delta.$$

The estimates (63) and (65) yield

$$(66) \quad \begin{aligned} \|\mathbf{Q}\| &= \|\mathbf{W} \cdot \mathbf{U}^{-1}\| \leq \|\mathbf{W}\| \cdot \|\mathbf{U}^{-1}\| \leq \{2\varepsilon\sqrt{n}\} \cdot \|\mathbf{U}^{-1}\| \\ &\leq \{2\varepsilon\sqrt{n}\} \cdot \{n(2e_2)^{n-1}/\delta\} = \varepsilon \cdot 2^n n^{3/2} (e_2)^{n-1}/\delta. \end{aligned}$$

COROLLARY 11.5. *Assume that $\varepsilon < \delta/2^{n+1}n^{3/2}(e_2)^{n-1}$, then $\|\mathbf{Q}\| < 1/2$ and so \mathbf{V}^{-1} exists. Moreover, we have the estimate $\|\mathbf{V}^{-1}\| \leq n \cdot 2^n (e_2)^{n-1}/\delta$.*

Proof. This follows from Lemma 11.2 and Corollary 11.4. \square

11.4. Estimate on perturbation of the center of an inscribed sphere. Our next goal is to obtain an effective estimate on $\|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\|$. For that we estimate the remaining terms in the equations (56) and (57). By (61) and (62),

$$(67) \quad \|\vec{\lambda}(\mathbf{V})\| \leq (2e_3)^2\sqrt{n} \quad , \quad \|\vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U})\| \leq 4\varepsilon(e_2 + e_3)\sqrt{n}.$$

We now return to the task of estimating $\|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\|$. Consider:

$$(68) \quad \begin{aligned} &2 \|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\| \\ &= \left\| \left\{ \mathbf{V}^{-1} \cdot \vec{\lambda}(\mathbf{V}) \right\} + 2\vec{z}_n - \left\{ \mathbf{U}^{-1} \cdot \vec{\lambda}(\mathbf{U}) \right\} - 2\vec{y}_n \right\| \\ &= \left\| 2(\vec{z}_n - \vec{y}_n) + \left\{ \mathbf{U}^{-1}(\mathbf{I} + \mathbf{Q})^{-1} \cdot \vec{\lambda}(\mathbf{V}) \right\} - \left\{ \mathbf{U}^{-1} \cdot \vec{\lambda}(\mathbf{U}) \right\} \right\| \\ &\leq 2\|\vec{z}_n - \vec{y}_n\| + \|\mathbf{U}^{-1}\| \cdot \left\| (\mathbf{I} - \mathbf{Q} + \mathbf{Q}^2 - \mathbf{Q}^3 \pm \dots) \cdot \vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U}) \right\| \\ &\leq 2\|\vec{z}_n - \vec{y}_n\| + \|\mathbf{U}^{-1}\| \cdot \left\{ \|\vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U})\| + \|(-\mathbf{Q} + \mathbf{Q}^2 - \mathbf{Q}^3 \pm \dots) \cdot \vec{\lambda}(\mathbf{V})\| \right\} \\ &\leq 2\|\vec{z}_n - \vec{y}_n\| + \|\mathbf{U}^{-1}\| \cdot \left\{ \|\vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U})\| + \|\vec{\lambda}(\mathbf{V})\| \cdot \|\mathbf{Q}\|/(1 - \|\mathbf{Q}\|) \right\}. \end{aligned}$$

Assume that $\varepsilon < \delta/2^{n+1}n^{3/2}(e_2)^{n-1}$, hence $\|\mathbf{Q}\| < 1/2$ by Corollary 11.5 and so $\|\mathbf{Q}\|/(1 - \|\mathbf{Q}\|) < 1$ and thus \mathbf{V}^{-1} exists. We use the more accurate estimate $\|\mathbf{Q}\| \leq \varepsilon 2^n n^{3/2} (e_2)^{n-1} / \delta$ from (66) which, combined with the previous estimates $\|\vec{y}_n - \vec{z}_n\| < \varepsilon$, (65) and (67), then (68) becomes

$$\begin{aligned} & 2 \|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\| \\ & \leq 2\|\vec{z}_n - \vec{y}_n\| + \|\mathbf{U}^{-1}\| \cdot \left\{ \|\vec{\lambda}(\mathbf{V}) - \vec{\lambda}(\mathbf{U})\| + \|\vec{\lambda}(\mathbf{V})\| \cdot \|\mathbf{Q}\|/(1 - \|\mathbf{Q}\|) \right\} \\ & \leq 2\varepsilon + \{n \cdot (2e_2)^{n-1}/\delta\} \cdot \{4\varepsilon(e_2 + e_3)\sqrt{n} + (2e_3)^2 \cdot 2\varepsilon \cdot 2^n n^2 (e_2)^{n-1}/\delta\}. \end{aligned}$$

Then using that $e_3 = e_2 + \varepsilon > e_2$ we have

$$(69) \quad \|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\| < \varepsilon \cdot \left\{ 1 + n^{3/2} 2^{n+1} (e_3)^n / \delta + 2n^3 2^{2n} (e_3)^{2n} / \delta^2 \right\}.$$

It is important to note that the ratios $(e_3)^n/\delta$ and $(e_3)^{2n}/\delta^2$ are “dimensionless”, so the estimate (69) is scale invariant, in that the expression in brackets on the right hand side is unchanged by scalar multiplication on \mathbb{R}^n .

11.5. Robustness of simplices. We next give an estimate for δ , the constant in (69) which is a lower bound on $|\mathbf{U}|$, or equivalently on the volume of the simplex $\Delta(\vec{y}_0, \dots, \vec{y}_n)$. Since the edges of the simplex have lengths bounded by $2e_2$, this condition guarantees that the vertex \vec{y}_k in a simplex is not too close to a $k - 1$ -dimensional subspace defined by $\{\vec{y}_0, \dots, \vec{y}_{k-1}\}$, and so ensures that a small perturbation of vertices does not change drastically the geometry of the simplicial complex.

DEFINITION 11.6. *Let $\rho > 0$ and $1 \leq m \leq n$. A collection of vectors $\{\vec{y}_0, \dots, \vec{y}_m\} \subset \mathbb{R}^n$ is said to be ρ -robust if for each $0 \leq k < m$, the distance from the point \vec{y}_{k+1} to the affine subspace spanned by the vertices $\{\vec{y}_0, \dots, \vec{y}_k\}$ is at least ρ .*

The significance of this definition is seen from an elementary estimation, whose proof follows by induction and standard Euclidean geometry. Let $P(\vec{y}_0, \dots, \vec{y}_n)$ denote the parallelepiped with edges $\vec{y}_i - \vec{y}_0$ for $1 \leq i \leq n$, and note that its volume is equal to $n! \cdot \text{Vol}(\Delta(\vec{y}_0, \dots, \vec{y}_n))$.

LEMMA 11.7. *Let $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$ be a ρ -robust collection, then $P(\vec{y}_0, \dots, \vec{y}_n)$ has volume at least $\rho^{n-1} \cdot \|\vec{y}_1 - \vec{y}_0\|$. \square*

This volume estimate can be improved when the vertices are lattice points on an inscribed sphere:

LEMMA 11.8. *For $0 < e_1 < e_2$, there exists $V_2(e_1, e_2) > 0$ such that given $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$, and $0 < r \leq e_2$ satisfying:*

- (1) $e_1 \leq \|\vec{y}_k - \vec{y}_j\|$ for $0 \leq j \neq k \leq n$,
- (2) $\|\vec{y}_k\| = r$ for all $0 \leq k \leq n$,
- (3) $\{\vec{y}_0, \dots, \vec{y}_n\}$ is ρ -robust.

Then $P(\vec{y}_0, \vec{y}_1, \dots, \vec{y}_n)$ has volume at least $V_2(e_1, e_2) \cdot \rho^{n-2}$.

Proof. First, note that the vectors $\{\vec{y}_0, \vec{y}_1, \vec{y}_2\} \subset \mathbb{R}^n$ cannot be collinear, as they lie on a sphere of radius $r \leq e_2$. Also, the vectors $\vec{\sigma}_1 = \vec{y}_1 - \vec{y}_0$ and $\vec{\sigma}_2 = \vec{y}_2 - \vec{y}_0$ have lengths greater than e_1 by (11.8.1), and thus define a non-degenerate parallelogram $P(\vec{y}_0, \vec{y}_1, \vec{y}_2)$. The minimum for the area over all such parallelograms must be positive, as these conditions define a compact set of such, all of which have positive area. Let $V_2(e_1, e_2) > 0$ denote this minimum.

Next, the vector \vec{y}_3 lies at distance at least ρ from the plane spanned by $\{\vec{y}_0, \vec{y}_1, \vec{y}_2\}$ by the ρ -robust assumption. As \vec{y}_0 lies on this plane, $\vec{\sigma}_3 = \vec{y}_3 - \vec{y}_0$ must also lie distance at least ρ from it. Thus, $P(\vec{y}_0, \vec{y}_1, \vec{y}_2, \vec{y}_3)$ with edges by $\{\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3\}$ has 3-volume bounded below by $V_2(e_1, e_2) \cdot \rho$.

Continuing by induction, one has that the parallelepiped $P(\vec{y}_0, \vec{y}_1, \dots, \vec{y}_k)$ with edges $\{\vec{\sigma}_1, \dots, \vec{\sigma}_k\}$ has k -volume bounded below by $V_2(e_1, e_2) \cdot \rho^{k-2}$ for all $2 < k \leq n$. \square

Lemma 11.8 hints at a fundamental difference between the study of Delaunay triangulations in dimension 2, and the theory for dimensions greater than two. The volume estimate for simplices in dimension two admits a uniform lower positive bound depending only on the constants $0 < e_1 < e_2$. For higher dimensions, there is an additional restriction required to obtain an estimate, the *robustness* of the vertices, or some equivalent version of this condition. For example, if bounds are given on the interior angles of the simplex, then this observation is surely well known.

We combine the above results to obtain the final form (70) of the desired estimate:

PROPOSITION 11.9. *Let $\{\vec{y}_0, \dots, \vec{y}_n\} \subset \mathbb{R}^n$ be $\rho > 0$ robust, and admit an inscribed sphere with center $\omega(\vec{y}_0, \dots, \vec{y}_n)$ and radius $r(\vec{y}_0, \dots, \vec{y}_n)$. Given $0 < e_1 < e_2$, set $\delta = V_2(e_1, e_2) \cdot \rho^{n-2}$, and let $\varepsilon > 0$. Suppose that, in addition, we have:*

- (1) $e_1 \leq \|\vec{y}_i - \vec{y}_j\|$ for all $0 \leq i \neq j \leq n$,
- (2) $e_1/2 \leq r(\vec{y}_0, \dots, \vec{y}_n) \leq e_2$,
- (3) $\varepsilon \leq \delta/2^{n+1} n^{3/2} (e_2)^{n-1} \leq \frac{V_2(e_1, e_2) \cdot \rho^{n-2}}{2^{n+1} n^{3/2} (e_2)^{n-1}}$.

Let $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$ satisfy

- (4) $\|\vec{y}_i - \vec{z}_i\| \leq \varepsilon$ for all $0 \leq i \leq n$,

then $\{\vec{z}_0, \dots, \vec{z}_n\}$ has an inscribed sphere with center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ so that for $e_3 = e_2 + \varepsilon$,

$$(70) \quad \|\vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) - \vec{\omega}(\vec{y}_0, \dots, \vec{y}_n)\| < \varepsilon \cdot \left\{ 1 + n^{3/2} 2^{n+1} (e_3)^n / \delta + 2 n^3 2^{2n} (e_3)^{2n} / \delta^2 \right\}.$$

12. INSCRIBED SPHERES VIA INEQUALITIES

There is an alternative approach to showing the existence of an inscribed sphere for points $\{\vec{z}_0, \dots, \vec{z}_n\}$, based on being given an “approximate solution” to the problem defined by a system of inequalities. This approach is advantageous when considering perturbations of a given triangulation, and we develop some key estimates which are used later.

12.1. Approximating centers of inscribed spheres. Given vectors $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$, let \mathbf{V} denote the $n \times n$ matrix whose rows are the transposes of the vectors $\vec{v}_k = \vec{z}_{k-1} - \vec{z}_n$ for $1 \leq k \leq n$, and $\vec{\lambda}(\mathbf{V}) = (\|\vec{v}_1\|^2, \dots, \|\vec{v}_n\|^2)^t$. Assuming that \mathbf{V} is invertible, the first result gives an estimate on the distance between an approximate center for the points and the actual center.

PROPOSITION 12.1. *Suppose that we are given vectors $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$, $\omega \in \mathbb{R}^n$ and constants $0 < C_1 < r$ and $C_2 > 0$ such that*

- (1) $r - C_1 < \|\vec{z}_k - \omega\| < r + C_1$ for all $0 \leq k \leq n$,
- (2) $\|\mathbf{V}^{-1}\| \leq C_2$.

Then $\{\vec{z}_0, \dots, \vec{z}_n\}$ has an inscribed sphere with center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ such that

$$(71) \quad \|\omega - \omega(\vec{z}_0, \dots, \vec{z}_n)\| < 2\sqrt{n} \cdot r C_1 C_2$$

Proof. The existence of the inscribed sphere follows as before, given that \mathbf{V} is invertible. The center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ lies in the common intersection of the hyperplanes

$$\begin{aligned} L(\vec{z}_{k-1}, \vec{z}_n) &= \{\vec{x} \in \mathbb{R}^n \mid (\vec{z}_{k-1} - \vec{z}_n) \bullet (\vec{x} - (\vec{z}_{k-1} + \vec{z}_n)/2) = 0\} \\ &= \{\zeta + \vec{z}_n \mid \vec{v}_k \bullet \vec{\zeta} = 1/2 \cdot \|\vec{v}_k\|^2\} \end{aligned}$$

where $\zeta = \vec{x} - \vec{z}_n$. Thus, the solution of the matrix equation (57),

$$(72) \quad \mathbf{V} \cdot \zeta = \frac{1}{2} \cdot \vec{\lambda}(\mathbf{V}) \quad , \quad \vec{\omega}(\vec{z}_0, \dots, \vec{z}_n) = \frac{1}{2} \cdot \left\{ \mathbf{V}^{-1} \cdot \vec{\lambda}(\mathbf{V}) \right\} + \vec{z}_n$$

is the center for an inscribed sphere containing the points $\{\vec{z}_0, \dots, \vec{z}_n\}$. We must estimate $\|\omega'\|$ where $\omega' = \omega - \omega(\vec{z}_0, \dots, \vec{z}_n)$.

As $r - C_1 > 0$, the vector ω satisfies the inequalities

$$(73) \quad (r - C_1)^2 < \|\vec{z}_k - \omega\|^2 < (r + C_1)^2$$

Make the change of variables $\vec{v}_k = \vec{z}_{k-1} - \vec{z}_n$ and $\zeta = \omega - \vec{z}_n$, then expand to obtain

$$(74) \quad r^2 - 2rC_1 + C_1^2 < \|\vec{v}_k - \zeta\|^2 < r^2 + 2rC_1 + C_1^2$$

Note that $\vec{v}_{n+1} = \vec{z}_n - \vec{z}_n = \vec{0}$. Subtracting the inequalities (73) for $k = n + 1$ from those for $1 \leq k \leq n$ and expanding and canceling terms then yields

$$\begin{array}{llll} -4rC_1 & < & (\vec{v}_k - \zeta) \bullet (\vec{v}_k - \zeta) - (\vec{v}_{n+1} - \zeta) \bullet (\vec{v}_{n+1} - \zeta) & < & 4rC_1 \\ -4rC_1 & < & (\vec{v}_k \bullet \vec{v}_k - 2\vec{v}_k \bullet \zeta + \zeta \bullet \zeta) - \zeta \bullet \zeta & < & 4rC_1 \\ -4rC_1 & < & \vec{v}_k \bullet \vec{v}_k - 2\vec{v}_k \bullet \zeta & < & 4rC_1 \end{array}$$

Condition (12.1.1) and the above implies that $\zeta = \omega - \vec{z}_n$ is a solution of the matrix inequality

$$(75) \quad \mathbf{V} \cdot \zeta - \frac{1}{2} \vec{\lambda}(\mathbf{V}) \in B(0, 2\sqrt{n} \cdot rC_1)$$

The equation (72) implies that $\omega(\vec{z}_0, \dots, \vec{z}_n) - \vec{z}_n$ is a solution to the equation

$$(76) \quad \mathbf{V} \cdot \zeta - \frac{1}{2} \vec{\lambda}(\mathbf{V}) = \vec{0}$$

Thus, $\omega' = \omega - \omega(\vec{z}_0, \dots, \vec{z}_n)$ is a solution of the matrix inequality

$$(77) \quad \mathbf{V} \cdot \omega' \in B(0, 2\sqrt{n} \cdot rC_1)$$

We are given that $\|\mathbf{V}^{-1}\| \leq C_2$ hence we obtain the estimate (71). \square

12.2. Stability of Delaunay triangulations. The stability of the Delaunay triangulation associated to a net $\mathcal{N} \subset \mathbb{R}^n$ under perturbation of \mathcal{N} is equivalent to the stability of the inscribed spheres for the vertices of a simplex. The following result shows the existence of inscribed spheres based on estimates which are almost “stable under sufficiently small” perturbation.

PROPOSITION 12.2. *Let $\{\vec{z}_0, \dots, \vec{z}_n\} \subset \mathbb{R}^n$ be ρ -robust, for $\rho > 0$. Assume there are constants $0 < e_1 < e_2$ and $0 < C_1 < r < e_1$, and that there exists $\omega \in \mathbb{R}^n$ such that*

- (1) $e_1 < \|\vec{z}_i - \vec{z}_j\| < 2e_2$ for all $0 \leq i \neq j \leq n$
- (2) $r - C_1 < \|\vec{z}_k - \omega\| < r + C_1$ for all $0 \leq k \leq n$,

Then $\{\vec{z}_0, \dots, \vec{z}_n\}$ has an inscribed sphere with center $\omega(\vec{z}_0, \dots, \vec{z}_n)$ so that for

$$(78) \quad \|\omega - \omega(\vec{z}_0, \dots, \vec{z}_n)\| \leq C_1 \cdot n^{3/2} (2e_2)^{n-1} / \rho^{n-1}$$

Proof. Lemma 11.7 implies that the volume of the parallelepiped $P(\vec{z}_0, \dots, \vec{z}_n)$ with edges $\{\vec{v}_1, \dots, \vec{v}_n\}$ is bounded below by $e_1 \rho^{n-1}$, and hence $|\mathbf{V}| \geq e_1 \rho^{n-1}$. Thus by Corollary 11.4, we have

$$(79) \quad \|\mathbf{V}^{-1}\| \leq n(2e_2)^{n-1} / |\mathbf{V}| \leq n \cdot (2e_2)^{n-1} / e_1 \rho^{n-1}$$

Then (78) follows from estimate (71) of Proposition 12.1 and the hypotheses $r \leq e_1$. \square

Propositions 11.9 and 12.2 show the importance of the robustness condition in Definition 11.6 for estimating the stability of solutions for the equations (56). Our next result shows that a small perturbation of a robust simplex is also robust.

PROPOSITION 12.3. *Let $1 \leq m \leq n$, and assume that $\{\vec{y}_0, \dots, \vec{y}_m\} \subset \mathbb{R}^n$ is ρ -robust. Let $\{\vec{z}_0, \dots, \vec{z}_m\} \subset \mathbb{R}^n$ be also given, along with the constants $0 < e_1 < e_2$ and $0 < \varepsilon < e_1/4$ such that*

- (1) $e_1 \leq \|\vec{y}_i - \vec{y}_j\| \leq 2e_2$ for all $0 \leq i \neq j \leq m$
- (2) $\|\vec{y}_i - \vec{z}_i\| \leq \varepsilon$ for all $0 \leq i \leq m$.

Then $\{\vec{z}_0, \dots, \vec{z}_m\}$ is ρ_m -robust, for $\rho_m = \rho_m(\rho, \varepsilon, e_1, e_2)$ as defined below. Moreover, $\rho_m(\rho, \varepsilon, e_1, e_2)$ is monotone increasing in e_2 and ρ , and monotone decreasing in e_1 and ε , and is scale-invariant. That is, for $s > 0$, $\rho_m(s \cdot \rho, s \cdot \varepsilon, s \cdot e_1, s \cdot e_2) = s \cdot \rho_m(\rho, \varepsilon, e_1, e_2)$.

Proof. Set $e'_1 = e_1 - 2\varepsilon$, $e'_2 = e_2 + \varepsilon$ and $e_4 = 4(e_2 + e_1)$. Then for all $0 \leq i \neq j \leq m$,

$$e_1/2 < e'_1 < \|\vec{z}_i - \vec{z}_j\| < 2e'_2 < e_4$$

For each $0 \leq k \leq m$, let $\text{Span}(\vec{y}_0, \dots, \vec{y}_k) \subset \mathbb{R}^n$ denote the affine subspace spanned by the vectors, and let $\xi_k \in \text{Span}(\vec{y}_0, \dots, \vec{y}_{k-1})$ be the point closest to \vec{y}_k . Then $\rho \leq \|\vec{y}_k - \xi_k\| \leq \|\vec{y}_k - \vec{y}_0\| \leq 2e_2$.

Similarly, let $\text{Span}(\vec{z}_0, \dots, \vec{z}_{k-1}) \subset \mathbb{R}^n$ denote the affine subspace spanned by the vectors, and $\zeta_k \in \text{Span}(\vec{z}_0, \dots, \vec{z}_{k-1})$ be the point closest to \vec{z}_k . Then $\|\vec{z}_k - \zeta_k\| \leq \|\vec{z}_k - \vec{z}_j\| < 2e'_2$ for $j \leq k-1$.

The triangle inequality yields a lower bound

$$(80) \quad \begin{aligned} d_{\mathbb{R}^n}(\vec{z}_k, \text{Span}(\vec{z}_0, \dots, \vec{z}_{k-1})) = \|\vec{z}_k - \zeta_k\| &\geq \|\vec{y}_k - \xi_k\| - \|\vec{z}_k - \vec{y}_k\| - \|\xi_k - \zeta_k\| \\ &\geq \rho - \varepsilon - \|\xi_k - \zeta_k\| \end{aligned}$$

We develop an upper bound estimate for $\|\xi_k - \zeta_k\|$.

For the case $k = 1$, note that $\text{Span}(\vec{z}_0) = \{\vec{z}_0\}$ is just the single point, so $\xi_1 = \vec{y}_0$ and $\zeta_1 = \vec{z}_0$, and $\|\xi_1 - \zeta_1\| = \|\vec{z}_0 - \vec{y}_0\| \leq \varepsilon$, so in terms of the estimate (80) we have $d_{\mathbb{R}^n}(\vec{z}_1, \text{Span}(\vec{z}_0)) \geq \rho - 2\varepsilon$. Set $\delta_1 = 2$, then $\rho_1 = \rho - \varepsilon \cdot \delta_1$. This completes the proof of Proposition 12.3 for the case $m = 1$.

When $m > 1$ and $2 \leq k \leq m$, an upper bound estimate on $\|\xi_k - \zeta_k\|$ requires more delicate arguments.

We are given that $\vec{y}_j, \vec{z}_j \in D_{\mathbb{R}^n}(\vec{y}_k, 2e_2 + \varepsilon)$ for each $0 \leq j \leq m$. Since the distance from \vec{y}_k to ξ_k is at most that from \vec{y}_k to \vec{y}_0 we also have $\xi_k \in D_{\mathbb{R}^n}(\vec{y}_k, 2e_2)$. The analogous estimate is true for $d_{\mathbb{R}^n}(\vec{z}_k, \zeta_k)$, and since $\|\vec{y}_k - \vec{z}_k\| \leq \varepsilon$ we have that $\zeta_k \in D_{\mathbb{R}^n}(\vec{y}_k, 2e'_2)$. It follows that all of the points in consideration, $\vec{y}_j, \vec{z}_j, \xi_j, \zeta_j$, $1 \leq j \leq k$, lie in the closed disk $D_{\mathbb{R}^n}(\vec{y}_k, e_4)$ with radius $e_4 = 4(e_2 + e_1)$. This compactness estimate is fundamental.

Let $\text{Span}_k(\vec{y}_0, \dots, \vec{y}_{k-1}) = \text{Span}(\vec{y}_0, \dots, \vec{y}_{k-1}) \cap D_{\mathbb{R}^n}(\vec{y}_k, 2e'_2)$, the restricted subdisk of radius $2e'_2$. Note that we showed above that $\{\vec{y}_0, \dots, \vec{y}_{k-1}, \xi_1, \dots, \xi_k\} \subset \text{Span}_k(\vec{y}_0, \dots, \vec{y}_{k-1})$.

For the case $k = 2$, note that $\|\vec{y}_1 - \vec{y}_0\| \geq e_1$ and $\|\vec{z}_1 - \vec{z}_0\| \geq e'_1 > e_1/2$, and using that the disk $D_{\mathbb{R}^n}(\vec{y}_2, 2e'_2)$ has diameter at most e_4 , we have

$$(81) \quad \text{Span}_2(\vec{y}_0, \vec{y}_1) \subset \{\vec{y}_0 + t_1(\vec{y}_1 - \vec{y}_0) \mid -e_4/e_1 \leq t_1 \leq e_4/e_1\}$$

$$(82) \quad \text{Span}_2(\vec{z}_0, \vec{z}_1) \subset \{\vec{z}_0 + s_1(\vec{z}_1 - \vec{z}_0) \mid -e_4/e'_1 \leq s_1 \leq e_4/e'_1\}$$

LEMMA 12.4. *Given $\vec{z} \in \text{Span}_2(\vec{z}_0, \vec{z}_1)$, there exists $\vec{y} \in \text{Span}(\vec{y}_0, \vec{y}_1)$ so that*

$$(83) \quad \|\vec{z} - \vec{y}\| \leq \varepsilon \cdot (1 + 4e_4/e_1)$$

Proof. The point $\vec{z} \in \text{Span}_2(\vec{z}_0, \vec{z}_1)$ can be written as $\vec{z} = \vec{z}_0 + s_1 \cdot (\vec{z}_1 - \vec{z}_0) \in \text{Span}_2(\vec{z}_0, \vec{z}_1)$.

Then for $\vec{y} = \vec{y}_0 + s_1 \cdot (\vec{y}_1 - \vec{y}_0) \in \text{Span}_2(\vec{y}_0, \vec{y}_1)$ we have

$$\begin{aligned} \|\vec{z} - \vec{y}\| &= \|\{\vec{z}_0 + s_1 \cdot (\vec{z}_1 - \vec{z}_0)\} - \{\vec{y}_0 + s_1 \cdot (\vec{y}_1 - \vec{y}_0)\}\| \\ &\leq \|\vec{z}_0 - \vec{y}_0\| + |s_1| \|(\vec{z}_1 - \vec{z}_0) - (\vec{y}_1 - \vec{y}_0)\| \\ &\leq \varepsilon + |s_1|(\varepsilon + \varepsilon) \leq \varepsilon \cdot (1 + e_4/e'_1 \cdot 2) \leq \varepsilon \cdot (1 + 4e_4/e_1) \end{aligned}$$

Thus, every point of $\text{Span}_2(\vec{z}_0, \vec{z}_1)$ has distance at most $\varepsilon \cdot (1 + 4e_4/e_1)$ from a point of $\text{Span}(\vec{y}_0, \vec{y}_1)$. \square

Lemma 12.4 implies that $\|\xi_2 - \zeta_2\| \leq \varepsilon \cdot (1 + 4e_4/e_1)$, hence $\|\vec{z}_2 - \zeta_2\| \geq \rho_2$ by (80), where

$$(84) \quad \rho_2 = \rho - \varepsilon \cdot (2 + 4e_4/e_1) = \rho - \varepsilon \cdot \delta_2(\rho, e_1, e_2)$$

Note that $\delta_2(\rho, e_1, e_2) = (2 + 4e_4/e_1)$ depends only on the constants e_1 and e_2 , and as the ratio e_4/e_1 is scale invariant, thus ρ_2 is also scale invariant. If $m = 2$ then we are done.

Next, consider the case $k = 3$. The estimate ρ_3 in this case is obtained from (80) by subtracting from ρ a term which involves linear combinations of \vec{y}_2 with points of the line $\text{Span}(\vec{y}_0, \vec{y}_1)$, and the closer that \vec{y}_2 lies to this line, the larger the possible error, and likewise for $\text{Span}_3(\vec{z}_0, \vec{z}_1, \vec{z}_2)$.

As seen before for $k = 2$, the strategy is to estimate the parameters used to describe the planar region $\text{Span}_3(\vec{y}_0, \vec{y}_1, \vec{y}_2)$ as in (81), and similarly for $\text{Span}_3(\vec{z}_0, \vec{z}_1, \vec{z}_2)$ as in (82).

Recall that $\xi_2 \in \text{Span}(\vec{y}_0, \vec{y}_1)$ is the point on the line closest to \vec{y}_2 , and $\rho \leq \|\vec{y}_2 - \xi_2\| \leq 2e_2 < e_4$.

Likewise, the point $\zeta_2 \in \text{Span}(\vec{z}_0, \vec{z}_1)$ closest to \vec{z}_2 satisfies $\rho_2 \leq \|\vec{z}_2 - \zeta_2\| \leq 2e'_2 < e_4$.

Now let $\xi'_2 \in \text{Span}(\vec{y}_0, \vec{y}_1)$ be the point closest to ζ_2 . Then $\|\vec{y}_2 - \xi'_2\| \geq \|\vec{y}_2 - \xi_2\| \geq \rho > \rho_2$.

Furthermore, from the case $k = 2$, we have that $\|\xi'_2 - \zeta_2\| \leq \varepsilon \cdot \delta_2(\rho, e_1, e_2)$.

The key idea is to bound the space $\text{Span}_3(\vec{y}_0, \vec{y}_1, \vec{y}_2)$ using linear combinations with $(\vec{y}_2 - \xi'_2)$ and parameter bounds invoking ρ and ρ_2 :

$$\begin{aligned} \text{Span}_3(\vec{y}_0, \vec{y}_1, \vec{y}_2) &\subset \{ \vec{y}_0 + t_1(\vec{y}_1 - \vec{y}_0) + t_2(\vec{y}_2 - \xi'_2) \mid -e_4/e_1 \leq t_1 \leq e_4/e_1, -e_4/\rho \leq t_2 \leq e_4/\rho \} \\ \text{Span}_3(\vec{z}_0, \vec{z}_1, \vec{z}_2) &\subset \{ \vec{z}_0 + s_1(\vec{z}_1 - \vec{z}_0) + s_2(\vec{z}_2 - \zeta_2) \mid -e_4/e'_1 \leq s_1 \leq e_4/e'_1, -e_4/\rho_2 \leq s_2 \leq e_4/\rho_2 \} \end{aligned}$$

As in the proof of Lemma 12.4, every point of $\text{Span}_3(\vec{z}_0, \vec{z}_1, \vec{z}_2)$ thus lies a distance at most

$$\varepsilon \cdot \{1 + 2e_4/e_1 \cdot (1 + 1) + 2e_4/\rho_2 \cdot (1 + \delta_2)\}$$

from a point of $\text{Span}_3(\vec{y}_0, \vec{y}_1, \vec{y}_2)$, and in particular this estimate holds for $\|\xi_3 - \zeta_3\|$. Set

$$(85) \quad \delta_3 = \delta_3(\rho, e_1, e_2) = 2 + 4e_4/e_1 + (1 + \delta_2) \cdot 2e_4/\rho_2$$

Note that the ratio e_4/ρ_2 is scale-invariant, as is δ_2 , and thus δ_3 is scale-invariant.

Then for $\rho_3 = \rho - \varepsilon \cdot \delta_3$ by (80) we have $\|\vec{z}_3 - \zeta_3\| \geq \rho_3$.

Continuing in the way, given ρ_k and δ_k for $2 \leq k < m$, define inductively

$$(86) \quad \delta_{k+1} = 1 + \{1 + 2 \cdot 2e_4/e_1 + (1 + \delta_2) \cdot 2e_4/\rho_2 + \cdots + (1 + \delta_k) \cdot 2e_4/\rho_k\}$$

$$(87) \quad \rho_{k+1} = \rho - \varepsilon \cdot \delta_{k+1}$$

Then we have $\|\vec{z}_{k+1} - \zeta_{k+1}\| \geq \rho_{k+1}$. Continuing until $k + 1 = m$, we obtain

$$(88) \quad \delta_m(\rho, e_1, e_2) = 2 + 4e_4/e_1 + 2 \cdot \sum_{k=2}^{m-1} \frac{(1 + \delta_k)e_4}{\rho_k}$$

$$(89) \quad \rho_m(\rho, \varepsilon, e_1, e_2) = \rho - \varepsilon \cdot \delta_m(\rho, e_1, e_2)$$

for which $d_{\mathbb{R}^m}(\vec{z}_m, \text{Span}(\vec{z}_0, \dots, \vec{z}_{m-1})) = \|\vec{z}_m - \zeta_m\| \geq \rho_m(\rho, \varepsilon, e_1, e_2)$.

Observe that by the inductive definition (87), the values $\rho > \rho_1 > \cdots > \rho_m$ are monotone decreasing. Furthermore, by an inductive argument, for each $1 \leq k < m$ the value of ρ_k is a monotone increasing function of e_2 and ρ , and monotone decreasing for e_1 , and thus each term $(1 + \delta_k)e_4/\rho_k$ in the sum (88) is also monotone increasing, hence the same holds for $\rho_m(\rho, \varepsilon, e_1, e_2)$. Also note that each additional term $(1 + \delta_k) \cdot 2e_4/\rho_k$ in (86) is scale-invariant, so the sum (89) is scale-invariant. \square

The results of Part III give technical estimates on the construction of the Delaunay triangulation of an (e_1, e_2) -net in the Euclidean space \mathbb{R}^n , as developed in Part II, and also develop stability criteria for small perturbations of such a net. As such, the results should be compared to those in [35]. Our interest is in extending the results of Part III from Euclidean flat space, to coordinate charts on a Riemannian manifold which are sufficiently small so that they appear “almost flat”. This will be done in the next sections, in Part IV.

PART IV - MICRO-LOCAL RIEMANNIAN GEOMETRY

In Part IV, we discuss the “micro-local Riemannian geometry” of a matchbox manifold \mathfrak{M} . The goal is to extend several key concepts of Parts II and III to the leaves of \mathfrak{M} . This requires a sequence of estimates, the result of which is that the leafwise disks of a fixed radius can be assumed to be “ ϵ -approximately Euclidean”, and vary in the transverse direction by a controlled amount. We begin the development in Section 13, then pause to set a series of constants in Section 14 which are fundamental for the estimates. Finally, Section 15 gives the two main approximation results, Propositions 15.4 and 15.7, required for the constructions in Part V.

13. MICRO-LOCAL FOLIATION GEOMETRY

We are given, for each $1 \leq i \leq \nu$, the coordinate chart $\varphi_i: \overline{U}_i \rightarrow [-1, 1]^n \times \mathfrak{T}_i$, with notations as in Part II. Then for $x \in \overline{U}_i$, let $\mathcal{P}_i(x)$ be the plaque for the chart φ_i containing x . Moreover, we define $\lambda_i: \overline{U}_i \rightarrow [-1, 1]^n$ by setting $\varphi_i(x) = (\lambda_i(x), w_x) \in [-1, 1]^n \times \mathfrak{T}_i$. The map λ_i defines the smooth structure on each plaque $\mathcal{P}_i(x)$. Also, recall that $\lambda_{\mathcal{F}} > 0$ was chosen in Lemma 2.5 so that for all $x \in \mathfrak{M}$, the closed leafwise disk $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}})$ is strongly convex, and $2\delta_{\mathcal{U}}^{\mathcal{F}} < \lambda_{\mathcal{F}}/2$ bounds the diameter of the plaques in the foliation covering.

For $x \in \overline{U}_i$ define the transversal section, for $\mathfrak{U} \subset \mathfrak{T}_i$

$$(90) \quad \mathfrak{Z}(x, i, \mathfrak{U}) \equiv \varphi_i^{-1}(\lambda_i(x), \mathfrak{U}) ; \quad \mathfrak{Z}(x, i) \equiv \varphi_i^{-1}(\lambda_i(x), \mathfrak{T}_i) = \lambda_i^{-1} \circ \lambda_i(x)$$

As a special case, for $r \geq 0$, define the compact “disk section”

$$(91) \quad \mathfrak{Z}(x, i, r) \equiv \varphi_i^{-1}(\lambda_i(x), D_{\mathfrak{X}}(w_x, r) \cap \mathfrak{T}_i) \subset \overline{U}_i$$

The local coordinate charts $\varphi_i: \overline{U}_i \rightarrow [-1, 1]^n \times \mathfrak{T}_i$ are used to define a local “vertical translation” between plaques, which will be fundamental in the following. For $x' \in \mathfrak{Z}(x, i)$, define

$$(92) \quad \phi_i(x, x'): \mathcal{P}_i(x) \rightarrow \mathcal{P}_i(x') , \quad \xi' = \phi(x, x')(\xi) = \mathfrak{Z}(\xi, i) \cap \mathcal{P}_i(x')$$

When expressed in coordinates,

$$(93) \quad \varphi_i \circ \phi_i(x, x') \varphi_i^{-1}(\lambda_i(x), w_x) = (\lambda_i(x), w_{x'})$$

which is just the constant map in the first coordinate. Thus $\phi_i(x', x'') \circ \phi_i(x, x') = \phi_i(x, x'')$, and the maps $\phi_i(x, x')$ are homeomorphisms which depend continuously on $x' \in \mathfrak{T}_i$ in the C^0 -topology.

13.1. Metric distortions in coordinate charts. The construction of a stable nice transversal for \mathfrak{N} is based on the construction of nice Delaunay triangulations of a compact subset of families of leaves, which have strong invariance properties with respect to the maps $\phi_i(x, x')$ for the coordinate charts of \mathfrak{M} . To obtain the stability of these triangulations, as used in Section 10, the construction we give in the Section 16 requires very fine control on the metric distortions of the transverse translations $\phi_i(x, x')$. We make these requirements precise.

13.1.1. Leafwise metric distortions. We introduce estimates on the *leafwise metric* distortions of the maps $\phi_i(x, x')$. First, compare the Riemannian distance functions induced on differing plaques in the same chart \overline{U}_i by defining:

$$(94) \quad \begin{aligned} \text{var}(i, r) &= \max \{ |d_{\mathcal{F}}(x, y) - d_{\mathcal{F}}(x', y')| \mid x \in \overline{U}_i, x' \in \mathfrak{Z}(x, i, r), y \in \mathcal{P}_i(x), y' = \phi(x, x')(y) \} \\ &= \max \{ \{ |d_{\mathcal{F}}(y, z) - d_{\mathcal{F}}(\phi_i(x, x')(y), \phi_i(x, x')(z))| \} \mid y, z \in \mathcal{P}_i(x), x' \in \mathfrak{Z}(x, i, r) \} \end{aligned}$$

Note that $\text{var}(i, r)$ depends continuously on r , that $\text{var}(i, 0) = 0$, and $\text{var}(i, r) \leq 2\delta_{\mathcal{U}}^{\mathcal{F}}$ as $\mathcal{P}_i(x)$ is contained in a disk in L_x of radius $\delta_{\mathcal{U}}^{\mathcal{F}}$.

13.1.2. *Variation of chart coordinate systems.* There is another measure of the metric distortion between plaques, this time in terms of the variation due to differing coordinate systems. For $z \in \overline{U}_i \cap \overline{U}_j$ we obtain two standard transversals $\mathfrak{Z}(z, i, r)$ and $\mathfrak{Z}(z, j, r)$ in \mathfrak{M} through z . Define the *divergence* between these two transversals by

$$(95) \quad \text{div}(z, i, j, r) = \max \{d_{\mathcal{F}}(x', y') \mid x' \in \mathfrak{Z}(z, i, r), y' \in \mathfrak{Z}(z, j, r), \mathcal{P}_i(x') \cap \mathcal{P}_j(y') \neq \emptyset\}$$

The assumption $\delta_{\mathcal{U}}^{\mathcal{F}} < \lambda_{\mathcal{F}}/4$ implies that $\text{div}(z, i, j, r) < \lambda_{\mathcal{F}}$. Note that $\text{div}(z, i, i, r) = 0$ and that $\text{div}(z, i, j, 0) = 0$. Define

$$(96) \quad \text{div}(z, r) = \max \{\text{div}(z, i, j, r) \mid z \in \overline{U}_i \cap \overline{U}_j\}$$

The condition $\mathcal{P}_i(x') \cap \mathcal{P}_j(y') \neq \emptyset$ is closed in x', y' , and hence $\text{div}(z, r)$ is an upper semi-continuous function of both z and r . In terms of the transverse translation maps ϕ_i , for $\varepsilon = \text{div}(z, r)$, the condition (96) implies that the compositions $\phi_i(x', z) \circ \phi_j(z, y')$ are ε -close to the identity.

13.1.3. *Geodesic coordinate systems adapted to the Riemannian metric.* The inductive construction of a Delaunay triangulation for a manifold of non-zero curvature is based on having locally linear approximations to “small disks” in the Riemannian manifold, as provided by selected local geodesic coordinate systems. A basic issue then, is to estimate the distortion in sufficiently small leafwise disks for geodesic coordinates, due to the variation of the Riemannian metric between leaves.

Let $\hat{e} \equiv \{\vec{e}_1, \dots, \vec{e}_n\}$ denote the standard orthonormal basis of \mathbb{R}^n . A point $\vec{x} \in \mathbb{R}^n$ is then written in coordinates as $\vec{a} = (a_1, \dots, a_n)$, where $\vec{x} = \hat{e} \cdot \vec{a} = a_1 \vec{e}_1 + \dots + a_n \vec{e}_n$. Recall that the closed ball of radius λ about the origin in the standard metric is denoted by $D(\lambda)$, or $D_{\mathbb{R}^n}(\lambda)$ when it is better to emphasize that the disk is defined using the standard norm $\|\cdot\| = \|\cdot\|_{\mathbb{R}^n}$.

For $x \in \mathfrak{M}$, and coordinate system φ_i with $x \in U_i$ the basis \hat{e} of \mathbb{R}^n defines a framing \hat{e}_w of $T_{\hat{0}}(-1, 1)^n \times \{w\}$ for each $w \in \mathfrak{T}_i$. For each $x \in U_i$, the differential of the coordinate map φ_i at x defines a linear isomorphism $d_x \varphi_i: T_x \mathcal{F} \cong \mathbb{R}^n$, by which \hat{e}_w induces a framing \hat{e}_x for $T_x \mathcal{F}$. If the curvature of leaves is non-zero, then the tangent map $d_x \varphi_i$ is typically not an isometry, and thus the framing \hat{e}_x is typically not orthonormal for the leafwise Riemannian metric.

The leafwise Riemannian metric on $T\mathcal{F}$ induces on each plaque $\mathcal{P}_i(w) = \varphi_i^{-1}((-1, 1)^n \times \{w\})$ of U_i a family of inner products on its tangent space, which in terms of the framing \hat{e}_x at $x \in \mathcal{P}_i(w)$ is denoted by the matrix $g_{jk}(x)$. By Theorem 2.3, the tensor $g_{jk}(x)$ varies continuously in $w \in \mathfrak{T}_i$ for the C^∞ -topology on functions on $\mathcal{P}_i(w)$.

Given an arbitrary orthonormal frame $\hat{u} = \{\vec{u}_1, \dots, \vec{u}_n\} \subset T_x \mathcal{F}$ for the leafwise Riemannian metric, define a linear isomorphism

$$(97) \quad F_{\hat{u}}: \mathbb{R}^n \rightarrow T_x \mathcal{F} \cong \mathbb{R}^n, \quad F_{\hat{u}}(a_1, \dots, a_n) = \hat{u} \cdot \vec{a}$$

where we adopt the “matrix notation” $\hat{u} \cdot \vec{a} \equiv a_1 \vec{u}_1 + \dots + a_n \vec{u}_n \in T_x \mathcal{F}$. Via the coordinate isomorphism $d_x \varphi_i$, the tangent vectors \vec{u}_k form an orthonormal set $\hat{u} \subset \mathbb{R}^n$ for the inner product $g_{jk}(x)$, and in this sense, $\hat{u} \cdot \vec{a}$ is precisely a matrix product. To simplify notation, we let $\hat{u} = \hat{u}$ also denote this framing, as it is clear from context whether we consider the framing as in $T_x \mathcal{F}$ or in \mathbb{R}^n . Then $F_{\hat{u}}$ is a linear isometry between $\{\mathbb{R}^n, \|\cdot\|\}$ and $\{\mathbb{R}^n, \|\cdot\|_{\hat{u}}\}$, where $\|\cdot\|_{\hat{u}}$ denotes the norm on $T_x \mathcal{F} \cong \mathbb{R}^n$ induced by the inner product $g_{ij}(x)$.

Recall that $\exp_x^{\mathcal{F}}: T_x \mathcal{F} \rightarrow L_x$ is the leafwise geodesic map at x . Given an orthonormal framing \hat{u} of $T_x \mathcal{F}$ and $0 < \lambda \leq \lambda_{\mathcal{F}}$, the *leafwise geodesic coordinates* at x are defined by

$$(98) \quad \psi_{x, \hat{u}}^g: D_{\mathbb{R}^n}(\lambda) \rightarrow D_{\mathcal{F}}(x, \lambda) \subset L_x, \quad \psi_{x, \hat{u}}^g(\vec{a}) = \exp_x^{\mathcal{F}}(\hat{u} \cdot \vec{a})$$

Assume that $D_{\mathcal{F}}(x, \lambda) \subset U_i$, and let $\tilde{x} = \lambda_i(x) \in (-1, 1)^n \subset \mathbb{R}^n$. Then we have a second coordinate system on the neighborhood $D_{\mathcal{F}}(x, \lambda)$ of x , which is also “adapted” to the leafwise Riemannian metric on the disk $D_{\mathcal{F}}(x, \lambda)$. Define $T_{\tilde{x}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T_{\tilde{x}}(\vec{y}) = \tilde{x} + \vec{y}$, and compose $T_{\tilde{x}}$ with the framing map $F_{\hat{u}}$ to obtain:

$$(99) \quad \psi_{x, \hat{u}}^i \equiv \varphi_i^{-1}(T_{\tilde{x}} \circ F_{\hat{u}}, w_x): D_{\mathbb{R}^n}(\lambda) \rightarrow \mathcal{P}_i(x), \quad \psi_{x, \hat{u}}^i(\vec{y}) = \varphi_i^{-1}(\tilde{x} + \hat{u} \cdot \vec{y}, w_x)$$

Then $\psi_{x, \hat{u}}^i$ is the geodesic coordinate system for the *flat metric* on $\mathcal{P}_i(x)$ associated to $\|\cdot\|_{\hat{u}}$.

13.2. Comparison of geodesic coordinate systems. We compare the *affine geometries* defined by these two sets of coordinates, $\psi_{x,\hat{u}}^g$ and $\psi_{x,\hat{u}}^i$, using the coordinate system φ_i to convert the comparison to a local problem on \mathbb{R}^n involving differential equations on \mathbb{R}^n .

Recall that $D(\lambda) = D_{\mathbb{R}^n}(\lambda)$ is the Euclidean disk centered at the origin, $\tilde{x} = \lambda_i(x)$, and $D_{\tilde{g}}(\tilde{x}, s)$ denotes the closed disk of radius s about \tilde{x} for the metric \tilde{g} .

Let $\tilde{D}_i(\tilde{x}, \lambda) = \varphi_i(D_{\mathcal{F}}(x, \lambda)) \subset (-1, 1)^n \times \{w_x\}$ denote the image of the disk in the leafwise metric.

Let \tilde{d} denote the distance function on $\tilde{D}_i(\tilde{x}, \lambda)$ defined by the leafwise metric $d_{\mathcal{F}}$. That is, for $\vec{y}, \vec{z} \in \tilde{D}_i(\tilde{x}, \lambda)$, $\tilde{d}(\vec{y}, \vec{z}) = d_{\mathcal{F}}(\varphi_i^{-1}(\vec{y}, w_x), \varphi_i^{-1}(\vec{z}, w_x))$.

Let \tilde{g} denote the metric tensor on $\tilde{D}_i(x, \lambda)$ in the coordinates φ_i . Note that the image under φ_i of a geodesic segment for g is a geodesic segment for \tilde{g} , and as $D_{\mathcal{F}}(x, \lambda)$ is strongly convex for $\lambda \leq \lambda_{\mathcal{F}}$, the same holds for the region $\tilde{D}_i(\tilde{x}, \lambda)$ with the metric \tilde{g} .

Let $\widetilde{\exp}_{\tilde{x}}$ denote the geodesic map associated to \tilde{g} , centered at \tilde{x} . Then for the orthonormal basis $\hat{u} \subset T_x \mathcal{F}$ considered as a frame for $T_{\tilde{x}} \mathbb{R}^n$, we set

$$(100) \quad \widetilde{\exp}_{\tilde{x}, \hat{u}}: D(\lambda) \rightarrow \tilde{D}_i(\tilde{x}, \lambda), \quad \widetilde{\exp}_{\tilde{x}, \hat{u}}(\vec{a}) = \widetilde{\exp}_{\tilde{x}}(\hat{u} \cdot \vec{a})$$

Recall that we also have a linear map $T_{\tilde{x}} \circ F_{\hat{u}}$, which is a linear isometry between $\{\mathbb{R}^n, \|\cdot\|\}$ and $\{\mathbb{R}^n, \|\cdot\|_{\hat{u}}\}$, and satisfies $T_{\tilde{x}} \circ F_{\hat{u}}(\vec{0}) = \tilde{x} = \widetilde{\exp}_{\tilde{x}, \hat{u}}(\vec{0})$. Let $g^{\hat{u}} = (T_{\tilde{x}} \circ F_{\hat{u}})^*(\tilde{g})$ denote the metric \tilde{g} near \tilde{x} pulled back to $D(\lambda)$ via the isometry $T_{\tilde{x}} \circ F_{\hat{u}}$. Then $g_{jk}^{\hat{u}}(\vec{a}) = \delta_{jk}$ for $\vec{a} = \vec{0}$ by definition of \hat{u} . The metric $g^{\hat{u}}$ is in *Gauss normal form* [13, 25], and its metric tensor $g_{jk}^{\hat{u}}$ consequently has further special properties in a neighborhood of $\vec{0}$.

DEFINITION 13.1. Let $x \in \mathfrak{M}$ and $0 < \lambda \leq \lambda_{\mathcal{F}}/2$. Assume that $D_{\mathcal{F}}(x, \lambda) \subset \mathcal{P}_i(x)$, and let \hat{u} be an orthonormal frame for $T_x \mathcal{F}$. For $\varepsilon > 0$, we say that $\psi_{x,\hat{u}}^g: D(\lambda) \rightarrow D_{\mathcal{F}}(x, \lambda)$ is ε -approximately Euclidean if the following hold (in the coordinate system φ_i):

- (1) For all $\vec{a} \in D(\lambda)$,
 - (101)
$$\|g_{jk}^{\hat{u}}(\vec{a}) - \delta_{jk}\| \leq \varepsilon/n^2$$
 - (2) For all $\vec{a} \in D(\lambda)$,
 - (102)
$$\tilde{d}(\widetilde{\exp}_{\tilde{x}, \hat{u}}(\vec{a}), T_{\tilde{x}} \circ F_{\hat{u}}(\vec{a})) \leq \varepsilon \cdot \|\vec{a}\|$$
 - (3) For a geodesic $\tilde{\sigma}: [0, 1] \rightarrow \tilde{D}_i(\tilde{x}, \lambda)$ in the \tilde{d} metric, with $\tilde{\sigma}(0) = \vec{y}_0$ and $\tilde{\sigma}(1) = \vec{y}_1$, set $\tilde{\tau}(t) = t \cdot (\vec{y}_1 - \vec{y}_0) + \vec{y}_0$, then
 - (103)
$$\tilde{d}(\tilde{\sigma}(t), \tilde{\tau}(t)) \leq \varepsilon \cdot \tilde{d}(\vec{y}_0, \vec{y}_1), \text{ for all } 0 \leq t \leq 1$$
 - (4) For $s \leq \lambda$, the Riemannian volume of leafwise disks satisfies
 - (104)
$$|\text{Vol}(D(s)) - \text{Vol}_{\tilde{g}}(D_{\tilde{g}}(\tilde{x}, s))| \leq \varepsilon \cdot s^n$$
- where Vol denotes the Euclidean volume and $\text{Vol}_{\tilde{g}}$ is the volume form for the metric \tilde{g} . More generally, given an open set $U \subset D(s)$, for $s \leq \lambda$, we require that
- (105)
$$|\text{Vol}(U) - \text{Vol}_{\tilde{g}}(\widetilde{\exp}_{\tilde{x}, \hat{u}}(U))| \leq \varepsilon \cdot s^n$$

Conditions (13.1.1) and (13.1.4) concern the continuity of the metric tensor \tilde{g} , while conditions (13.1.2-3) concern the behavior of geodesics for the metric \tilde{g} , so also require control on the first and second order derivatives of \tilde{g} . The condition (13.1.3) is simply that the geodesics for the metric \tilde{g} and the flat metric defined by \hat{u} “stay close”. The conditions (13.1.1-5) are closely related, but are formulated separately in the form they will be used later.

LEMMA 13.2. Assume that $\psi_{x,\hat{u}}^g: D(\lambda) \rightarrow D_{\mathcal{F}}(x, \lambda)$ is ε -approximately Euclidean. Then

$$(106) \quad |\tilde{d}(\vec{z}, \vec{y}) - \|\vec{z} - \vec{y}\|_{\hat{u}}| \leq \varepsilon \cdot \|\vec{z} - \vec{y}\|_{\hat{u}} \text{ for all } \vec{y}, \vec{z} \in \tilde{D}_i(\tilde{x}, \lambda)$$

Thus, conditions (13.1.1) and (13.1.2) yield, for all $\vec{a} \in D(\lambda)$,

$$(107) \quad \|\widetilde{\exp}_{\tilde{x}, \hat{u}}(\vec{a}) - T_{\tilde{x}} \circ F_{\hat{u}}(\vec{a})\|_{\hat{u}} \leq 2\varepsilon\lambda$$

Proof. Condition (13.1.1) and the estimate (53) imply the bound $\|g^{\widehat{u}} - \delta\|_{\widehat{u}} \leq \varepsilon$ on the matrix norm, from which (106) follows. Condition (107) then follows, as $\|\vec{a}\| \leq \lambda$. \square

13.3. Existence of approximately Euclidean charts. Here is the key technical result about coordinate charts in Gauss normal form, which is the basis for constructing nice stable transversals when the leafwise Riemannian metric for \mathcal{F} has non-zero curvature tensor.

PROPOSITION 13.3. *For all $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that for all $x \in \mathfrak{M}$ with $D_{\mathcal{F}}(x, \lambda_\varepsilon) \subset U_i$ and orthonormal frame \widehat{u} of $T_x \mathcal{F}$, the chart $\psi_{x, \widehat{u}}^g: D(\lambda_\varepsilon) \rightarrow L_x$ is ε -approximately Euclidean.*

Proof. The claim is that $\widetilde{\exp}_{\vec{x}, \widehat{u}}$ is well-approximated by the affine map $T_{\vec{x}} \circ F_{\widehat{u}}$ for λ_ε sufficiently small. This follows from standard facts about the geodesic charts for smooth metrics. (For example, see [25, Chapter 5].) The only novelty is that we use the continuity of the Riemannian metric and its derivatives as functions of $x \in \mathfrak{M}$ to obtain uniform estimates, for all $x \in \mathfrak{M}$. We briefly sketch the arguments.

Let $\widetilde{h} = \widetilde{h}_{jk}(\xi)$ denote the Riemannian tensor on $D(\lambda)$ induced from \widetilde{g} by the geodesic map $\widetilde{\exp}_{\vec{x}, \widehat{u}}$. Note that geodesic coordinates have the property that $\widetilde{h}_{jk}(\vec{0}) = \delta_{jk}$, the Dirac δ -function. Moreover, the Riemannian Christoffel symbols $\widetilde{\Gamma}_{jk}^\ell(\xi)$ of the metric \widetilde{h} also vanish at the origin.

The tensor $\widetilde{\Gamma}_{jk}^\ell(\xi)$ is $C^{\ell-1}$ -continuous as a function of the metric tensor in the C^ℓ topology, for $\ell \geq 1$, so the first derivatives of $\widetilde{\Gamma}_{ij}^k(\xi)$ vary continuously with the metric in the C^2 topology, hence its curvature tensor $\widetilde{R}(\xi)$ varies continuously in the C^2 -topology as well. Thus, by choosing $\lambda > 0$ sufficiently small, we can assume the quantities $\|\widetilde{h}_{jk}(\xi) - \delta_{jk}\|$ and $|\widetilde{\Gamma}_{jk}^\ell(\xi)|$ are arbitrarily small on the disk $D(\lambda)$, and moreover the norm of the curvature tensor $|\widetilde{R}(\xi)|$ is uniformly bounded.

Standard results of Riemannian geometry show that the second derivatives of the geodesic map $\widetilde{\exp}_{\vec{x}, \widehat{u}}$ at the origin are bounded by the norms of the Christoffel symbols $\widetilde{\Gamma}_{jk}^\ell(\xi)$, of their derivatives, and of the curvature terms $\widetilde{R}(\xi)$. (For example, see [25, Chapter 5, Remark 2.11].) Thus, given $\varepsilon' > 0$, there exists $\lambda_{x, \varepsilon'} > 0$ such that $\widetilde{\exp}_{\vec{x}, \widehat{u}}$ is ε' -close to its linear approximation $T_{\vec{x}} \circ F_{\widehat{u}}$ in the Euclidean norm on \mathbb{R}^n . This yields the estimate (102) of Definition 13.1.2.

The condition (103) of Definition 13.1.3 follows, as the local expressions of the Christoffel symbols $\widetilde{\Gamma}_{jk}^\ell$ are sufficiently small on $D(\lambda)$ and the quantities $|\widetilde{\Gamma}_{jk}^\ell|$ are uniformly bounded. Conditions (13.1.1) and (13.1.4-5) follow from the continuity of the metric tensor \widetilde{g} , as noted above.

For each $\varepsilon > 0$, choose $\lambda_\varepsilon > 0$ so that the conditions of Definition 13.1 holds for all $x \in \mathfrak{M}$, and any choice of orthonormal frame \widehat{u} for $T_x \mathcal{F}$.

There is one further subtlety, which is that the error estimates in formulas (102) and (103) are in terms of the leafwise distance function $d_{\mathcal{F}}$, while the error ε' above is in terms of the Euclidean norm $\|\cdot\|$ on $D(\lambda)$. Introduce the constant

$$\|d_{\mathcal{F}}\| = \max \left\{ \frac{d_{\mathcal{F}}(\psi_{x, \widehat{u}}^g(\vec{b}), \psi_{x, \widehat{u}}^g(\vec{a}))}{\|\vec{b} - \vec{a}\|}, \frac{\|\vec{b} - \vec{a}\|}{d_{\mathcal{F}}(\psi_{x, \widehat{u}}^g(\vec{b}), \psi_{x, \widehat{u}}^g(\vec{a}))} \mid x \in \mathfrak{M}, \widehat{u}, \vec{a} \neq \vec{b} \in D(\lambda_{\mathcal{F}}) \right\}$$

Given ε , let $\varepsilon' = \varepsilon/\|d_{\mathcal{F}}\|$ and choose λ_ε for as above for the error ε' . Thus, by the compactness of \mathfrak{M} and the continuity of the metric in the C^2 -norm, given $\varepsilon > 0$ there exists an $\lambda_\varepsilon > 0$, so that for all $x \in \mathfrak{M}$, for all $0 < \lambda \leq \lambda_\varepsilon$, and for all coordinate chart indices $1 \leq i \leq \nu$ with $D_{\mathcal{F}}(x, \lambda) \subset \mathcal{P}_i(x)$, the estimates (102) to (104) of Definition 13.1 are satisfied. \square

REMARK 13.4. If the leaves of \mathcal{F} are isometric to Euclidean space \mathbb{R}^n , such as when \mathcal{F} is defined by a free action of \mathbb{R}^n , then λ_ε may be chosen arbitrarily large. Otherwise, if the leaves of \mathcal{F} have large sectional curvatures and ε is small, then λ_ε may be quite small. One consequence of $\lambda_\varepsilon \ll \lambda_{\mathcal{F}}$ is that it forces the points in the leafwise nets constructed in section 16 to be *very closely* spaced.

14. SETTING THE CONSTANTS

Our ultimate “affine approximation” results are given by Propositions 15.4 and 15.7 in the next section, which extend Proposition 13.3 above. However, in order to state and prove these results, it is necessary to specify the “universal scale constant” $\varepsilon_0 > 0$ for which these results are valid. It is absolutely fundamental that these estimates provided by these propositions are independent of the choices made in their applications. That is, we must a priori define the constant ε_0 as well as error bounds ε_1 , ε_2 , ε_3 and ε_4 which arise. For this reason, in this section we prescribe these geometric constraints, and then make our construction using these fixed choices. The word “arcane” describes perfectly the process for choosing these constants. This section can be skipped at first reading if desired, and then consulted later, though the process of making these choices, and some of their implications as pointed out below, are a key part of the construction.

14.1. Number of inscribed spheres. The first constant to define is a very large number, based on the combinatorics of nets in regions of \mathbb{R}^n , which is the reason for role of the dimension number n in the following. It may be possible to give a much more refined value to the constant, but for our purposes, the following suffices. Set:

$$(108) \quad C_n = \frac{10^n!}{1!(10^n - 1)!} + \frac{10^n!}{2!(10^n - 2)!} + \cdots + \frac{10^n!}{n!(10^n - n)!} + \frac{10^n!}{(n+1)!(10^n - n - 1)!}$$

Given a finite subset $\Omega \subset D_{\mathcal{F}}(\xi, \lambda_{\mathcal{F}}^*)$ with *cardinality bounded above by* 10^n , then C_n is an upper bound for the number of distinct subsets of Ω consisting of at most $(n+1)$ -distinct points. In particular, C_n is an upper bound on the number of distinct n -simplices, defined by $(n+1)$ -vertices in Ω . Thus, C_n is an upper bound on the number of inscribed spheres for the set Ω .

14.2. Geometric constants. Next, introduce four additional “geometric constants”. The purpose of these choices is briefly indicated, and their precise roles will be apparent later. The constants are defined now, as it is fundamental that these can be chosen independent of later choices. The constants are “scale-invariant”, and in their applications are multiplied by the scale $\lambda_{\mathcal{F}}^*$ which is defined by (117) below.

The *width of the annular regions* appearing in Lemma 17.3 will be chosen bounded above by

$$(109) \quad \varepsilon_1 = 1/(C_n \cdot 1000n \cdot 100^n).$$

The *thickness of the rectangular regions* appearing in the robustness condition (153) will be chosen bounded above by

$$(110) \quad \varepsilon_2 = 1/(C_n \cdot 2000 \cdot 2^n).$$

The *translation distance of the centers of inscribed spheres* for perturbed vertices is bounded by

$$(111) \quad \varepsilon_3 = \varepsilon_1/10.$$

The constant ε_3 first appears in the statement and proof of Proposition 18.1. We repeatedly use the implication of this choice that $\varepsilon_3 < \varepsilon_1/4$. Set

The *error of the affine approximation* in Proposition 15.4 is bounded by a constant ε_4 . The value of this constant determines the recursive decrease in the robustness estimates in Propositions 12.3, 15.7 and 18.1. The value of ε_4 is defined by a recursive process, depending on the dimension n , which we recall from the proof of Proposition 12.3.

Proposition 12.3 gives a recursive definition for the functions $\rho_m(\rho, \varepsilon, e_1, e_2)$ for $1 \leq m \leq n$. As noted there, the function $\rho_m(\rho, \varepsilon, e_1, e_2)$ is monotone increasing in e_2 and ρ , and monotone decreasing in e_1 and ε , and satisfies $\rho_m(s \cdot \rho, s \cdot \varepsilon, s \cdot e_1, s \cdot e_2) = s \cdot \rho_m(\rho, \varepsilon, e_1, e_2)$ for $s > 0$. Moreover, for all $1 \leq m \leq n$, $\rho_m(\rho, 0, e_1, e_2) = \rho$. For the normalized values $e_1 = 1$, $e_2 = 2$, $e_4 = 4(e_2 + e_1) = 12$, and $\rho = \rho_0$, define functions $\rho_m(\rho_0, \varepsilon)$ recursively by

$$\rho_0(\rho_0, \varepsilon) = \rho_0, \quad \delta_1 = 2, \quad \rho_1(\rho_0, \varepsilon) = \rho_0 - 2\varepsilon, \quad \delta_2 = 50, \quad \rho_2(\rho_0, \varepsilon) = \rho - 50\varepsilon$$

and for $1 < m \leq n$, by

$$(112) \quad \rho_m(\rho_0, \varepsilon) = \rho_0 - \varepsilon \cdot \delta_m, \quad \delta_m = 50 + 24 \cdot \sum_{k=2}^{m-1} \frac{(1 + \delta_k)}{\rho_k(\varepsilon)}$$

Note that each $\rho_m(\varepsilon)$ is a continuous function of ε . Also, for fixed initial data (ρ_0, ε) , the sequence of values is monotone decreasing in m :

$$\rho_0 = \rho_0(\rho_0, \varepsilon) > \rho_1(\rho_0, \varepsilon) > \rho_2(\rho_0, \varepsilon) > \cdots > \rho_n(\rho_0, \varepsilon) > 0$$

At a key stage of the induction process, we introduce the following constants, for each $0 \leq k \leq n$:

$$\begin{aligned} \hat{\rho}_k &= (18 - 2k/3n) \cdot \varepsilon_2 \\ \hat{\rho}'_k &= (18 - (2k+1)/3n) \cdot \varepsilon_2 \end{aligned}$$

Then we have

$$(113) \quad 18\varepsilon_2 = \hat{\rho}_0 > \hat{\rho}'_0 > \hat{\rho}_1 > \hat{\rho}'_1 > \cdots > \hat{\rho}_n > \hat{\rho}'_n > \hat{\rho}_{n+1} > \hat{\rho}'_{n+1} > 15\varepsilon_2$$

Finally, choose $\varepsilon_4 > 0$ sufficiently small so that the following $2n+2$ inequalities hold, for $1 \leq k \leq n+1$:

$$(114) \quad \hat{\rho}_k > \rho_n(\hat{\rho}_k, 10\varepsilon_4) > \hat{\rho}'_k + \varepsilon_2/100$$

$$(115) \quad \hat{\rho}'_k > \rho_n(\hat{\rho}'_k, 10\varepsilon_4) > \hat{\rho}_{k+1} + \varepsilon_2/100$$

The full set of these inequalities are used in the proofs of Propositions 17.6 and 18.1, where they are multiplied by the scale $s = \lambda_{\mathcal{F}}^*/10$.

14.3. Error of transverse computations. Finally, ε_0 is the “basic error” appearing in almost every transverse translation calculation and estimate, so is restricted by multiple conditions. The following restrictions are informally summarized by saying “it is intuitively clear that there exists ε_0 *sufficiently small* so that all of these conditions are satisfied”. We make this intuition precise:

(116) Choose $\varepsilon_0 > 0$ which satisfies the following conditions:

- (1) $\varepsilon_0 < 1/2000$ – used in equations (154) and (155)
- (2) $\varepsilon_0 \leq 50n(2/5)^{n\varepsilon_1}$ – used in equation (146)
- (3) $\varepsilon_0 < \varepsilon_2/2000$ – used in equations (151) and (152) and in proof of Proposition 17.6
- (4) $\varepsilon_0 < \varepsilon_3/4$ – used in equations (158), (179), (186) and in proof of Proposition 18.1
- (5) $\varepsilon_0 < \varepsilon_1/2 < \varepsilon_1 - 2\varepsilon_3$ – used in (196)
- (6) $\varepsilon_0 < \varepsilon_3/2\{1 + 35n^{3/2} \cdot (4/15\varepsilon_2)^{n-1}\}$ – used in equation (176)
- (7) $\varepsilon_0 < \varepsilon_4/20$ – used in Proposition 15.4
- (8) $\varepsilon_0 < \delta_n(\varepsilon_4)/100$ for δ_n defined in Lemma 15.5

Note that the function δ_n in estimate (8) above, as defined in Lemma 15.5, is derived from linear algebra, and is independent of all other choices, so that ε_0 is well-defined.

14.4. Leafwise constants. Recall that λ_{ε_0} was defined in the proof of Proposition 13.3. Introduce the fundamental “leafwise” constant:

$$(117) \quad \lambda_{\mathcal{F}}^* = \min\{\delta_{\mathcal{U}}^{\mathcal{F}}, \lambda_{\mathcal{F}}/5, \lambda_{\varepsilon_0}, 1\}$$

which is the basic distance scale for all of our subsequent constructions, chosen so that the leafwise balls $D_{\mathcal{F}}(\xi, \lambda_{\mathcal{F}}^*)$ are “ ε_0 -approximately Euclidean”. For example, if the leaves of \mathcal{F} are isometric to Euclidean space \mathbb{R}^n , then $\lambda_{\mathcal{F}}^* = \min\{\delta_{\mathcal{U}}^{\mathcal{F}}, \lambda_{\mathcal{F}}/5, 1\}$. Otherwise, if the leaves of \mathcal{F} have large sectional curvatures, then $\lambda_{\mathcal{F}}^*$ may be quite small.

14.5. Variations of the metric in charts. Recall the definitions of the functions var in (94) and div in (96), and choose the “transverse” scale constant $r_* > 0$ so that $div(z, r_*) \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$ for all $z \in \mathfrak{M}$, and also $var(i, r_*) \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$ for all $1 \leq i \leq \nu$.

15. AFFINE DISTORTION ESTIMATES

We next give several applications of the choices in Section 14 for the “micro-local geometry” of a matchbox manifold, culminating in the proofs of Propositions 15.4 and 15.7.

The following notion and estimates will be used frequently.

DEFINITION 15.1. *Let $\{X, d_X\}$ and $\{Y, d_Y\}$ be metric spaces, and $\epsilon > 0$. A homeomorphism into $\phi: X \rightarrow Y$ is said to be an ϵ -isometry if*

$$(118) \quad d_X(x, x') - \epsilon \leq d_Y(\phi(x), \phi(x')) \leq d_X(x, x') + \epsilon \quad \text{for all } x, x' \in X$$

LEMMA 15.2. *For $x \in \mathfrak{M}$ and orthonormal frame \hat{u} for $T_x \mathcal{F}$, the geodesic normal coordinate map $\psi_{x, \hat{u}}^g: D(\lambda_{\mathcal{F}}^*/2) \rightarrow D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)$ is an $\varepsilon_0 \lambda_{\mathcal{F}}^*$ -isometry from the metric $\|\cdot\|$ to the metric $d_{\mathcal{F}}$.*

Proof. As the Euclidean disk $D(\lambda_{\mathcal{F}}^*/2)$ has diameter $\lambda_{\mathcal{F}}^*$, the claim then follows from the estimate $\|\hat{g}^{\hat{u}} - \delta\|_{\hat{u}} \leq \varepsilon_0$ as in the proof of Lemma 13.2. \square

Recall that the disk section $\mathfrak{Z}(y, i, r_*)$ of radius $r_* > 0$ was defined by formula (91)).

LEMMA 15.3. *Let $x \in U_i$ and $y \in \mathcal{P}_i(x) \cap U_j$ for some $1 \leq i, j \leq \nu$. Assume that $x' \in \mathfrak{Z}(x, i, r_*)$ and $y' = \mathfrak{Z}(y, j, r_*) \cap \mathcal{P}_i(x')$. Then*

$$(119) \quad d_{\mathcal{F}}(x, y) - 2\varepsilon_0 \lambda_{\mathcal{F}}^* \leq d_{\mathcal{F}}(x', y') \leq d_{\mathcal{F}}(x, y) + 2\varepsilon_0 \lambda_{\mathcal{F}}^*$$

If either $i = j$ or $x = y$, then a more strict estimate holds:

$$(120) \quad d_{\mathcal{F}}(x, y) - \varepsilon_0 \lambda_{\mathcal{F}}^* \leq d_{\mathcal{F}}(x', y') \leq d_{\mathcal{F}}(x, y) + \varepsilon_0 \lambda_{\mathcal{F}}^*$$

Proof. Let $y'' = \mathfrak{Z}(y, i, r_*) \cap \mathcal{P}_i(x')$. Then $d_{\mathcal{F}}(y', y'') \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$ by the definition of the divergence (96) and of r_* . Thus, $|d_{\mathcal{F}}(x', y'') - d_{\mathcal{F}}(x', y')| \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$ by the triangle inequality.

Then by the definition of the variation (94) and r_* we also have $|d_{\mathcal{F}}(x', y'') - d_{\mathcal{F}}(x, y)| \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$. The estimates (119) and (120) then follow. \square

15.1. Estimates for variations of affine geometries. We next derive estimates comparing the local *affine geometry* of geodesic coordinates in nearby plaques.

Let $x \in D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*) \subset U_i$ and $y \in D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)$ so that $D_{\mathcal{F}}(y, \lambda_{\mathcal{F}}^*/2) \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*)$.

Let $x' \in \mathfrak{Z}(x, i, r_*)$ and set $y' = \phi_i(x, x')(y)$. Choose orthonormal frames \hat{u} for $T_x \mathcal{F}$ and \hat{v}' for $T_{y'} \mathcal{F}$, with corresponding geodesic coordinates $\psi_{x, \hat{u}}^g$ and $\psi_{y', \hat{v}'}^g$. Consider the composition

$$(121) \quad \Psi'_{x, y'} \equiv (\psi_{y', \hat{v}'}^g)^{-1} \circ \phi_i(x, x') \circ \psi_{x, \hat{u}}^g \circ T_{\xi}: D(\lambda_{\mathcal{F}}^*) \rightarrow \mathbb{R}^n$$

where $\xi = (\psi_{x, \hat{u}}^g)^{-1}(y)$, and $T_{\xi}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the affine transformation $T_{\xi}(\vec{a}) = \vec{a} + \xi$. Then

$$\Psi'_{x, y'}(\vec{0}) = (\psi_{y', \hat{v}'}^g)^{-1} \circ \phi_i(x, x') \circ \psi_{x, \hat{u}}^g(\xi) = (\psi_{y', \hat{v}'}^g)^{-1} \circ \phi_i(x, x')(y) = (\psi_{y', \hat{v}'}^g)^{-1}(y') = \vec{0}$$

The map $\Psi'_{x, y'}$ compares two coordinate systems about the point y' : one is the translate of the geodesic coordinates $\psi_{x, \hat{u}}^g$ centered at x but restricted to a neighborhood of y in its domain, and the other is centered at the translated point y' . Each coordinate system defines an “affine structure” in a neighborhood of y' . The next result shows that $\Psi_{x, y'}$ can be made “almost the identity” by the proper choice of the framing \hat{v}' , so that these affine structures are arbitrarily close.

PROPOSITION 15.4. *There exists a choice of orthonormal frame \hat{v} for $T_{y'} \mathcal{F}$ so that*

$$(122) \quad \Psi_{x, y'} \equiv (\psi_{y', \hat{v}}^g)^{-1} \circ \phi_i(x, x') \circ \psi_{x, \hat{u}}^g \circ T_{\xi}: D(\lambda_{\mathcal{F}}^*/2) \rightarrow \mathbb{R}^n$$

is $\varepsilon_4 \lambda_{\mathcal{F}}^$ -close to the identity, for ε_4 chosen to satisfy (114) and (115).*

Proof. We are given $x \in D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*) \subset U_i$, $y \in D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)$ so that $D_{\mathcal{F}}(y, \lambda_{\mathcal{F}}^*/2) \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*)$, and $x' \in \mathfrak{Z}(x, i, r_*)$ and set $y' = \phi_i(x, x')(y)$. Also given are frames \hat{u} for $T_x \mathcal{F}$ and \hat{v}' for $T_{y'} \mathcal{F}$.

The idea of the proof is simple, in that we express both geodesic coordinate maps $\psi_{x, \hat{u}}^g$ and $\psi_{y', \hat{v}'}^g$ in the local coordinates φ_i , as in the proof of Proposition 13.3. The key point of the proof follows from some delicate linear algebra, used to choose the new framing \hat{v} , and then estimating the distortion of the geodesic coordinates for this frame.

Let $\varphi_i(x) = (\tilde{x}, w_x)$, $\varphi_i(y) = (\tilde{y}, w_y)$ and $\varphi_i(y') = (\tilde{y}', w_{y'})$ for $w_x, w_{y'} \in \mathfrak{T}_i$, where as before in Section 13.2, $\tilde{x} = \lambda_i(x)$ and $\tilde{y} = \lambda_i(y)$. By definition, $\phi_i(x, x')$ is the identity map when expressed in the coordinate system φ_i , so the assumption $y' = \phi_i(x, x')(y)$ implies $\tilde{y}' = \tilde{y}$.

We mention a point of notation established in Section 13 and used repeatedly below. The “tilde” notation, $\tilde{x} \in (-1, 1)^n$ for example, denotes the horizontal coordinates of a point or set; the “prime” notation denotes a point or set in the translated plaque $\mathcal{P}_i(y')$; while $\vec{v} \in \mathbb{R}^n$ denotes a vector in the vector space \mathbb{R}^n , typically obtained from the inverse of the geodesic coordinates ψ^g .

Set $d_2 = \lambda_{\mathcal{F}}^*/5$. The restriction $\phi_i(x, x'): D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2) \rightarrow \mathcal{P}_i(w_{y'})$ is an $\varepsilon_0 \lambda_{\mathcal{F}}^*$ -isometry by Lemma 15.3, and thus $\phi_i(x, x')(D_{\mathcal{F}}(y, 2d_2)) \subset D_{\mathcal{F}}(y', \lambda_{\mathcal{F}}^*/2)$. Indeed,

$$\phi_i(x, x')(D_{\mathcal{F}}(y, 2d_2)) \subset D_{\mathcal{F}}(y', 2\lambda_{\mathcal{F}}^*/5 + \varepsilon_0 \lambda_{\mathcal{F}}^*),$$

and since by assumption $\varepsilon_0 < 1/2000$ we have

$$2\lambda_{\mathcal{F}}^*/5 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* \leq 401/1000 \lambda_{\mathcal{F}}^* < \lambda_{\mathcal{F}}^*/2.$$

This implies that the composition (122) is well-defined. Recall from Section 13.2 that we denoted

$$\begin{aligned} \tilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2) &= \varphi_i(D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)) \subset (-1, 1)^n \times \{w_x\}, \\ \tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2) &= \varphi_i(D_{\mathcal{F}}(y', \lambda_{\mathcal{F}}^*/2)) \subset (-1, 1)^n \times \{w_{y'}\}. \end{aligned}$$

Recall from Section 13.2 that \tilde{d} denotes the distance function on $\tilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2)$ defined by the leafwise metric $d_{\mathcal{F}}$ via the coordinates φ_i , and \tilde{g} denotes the induced Riemannian metric on $\tilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2)$. The geodesic coordinates about a neighborhood of $\tilde{x} \in \mathbb{R}^n$ associated to \tilde{g} and \hat{u} , are denoted by $\widetilde{\exp}_{\tilde{x}, \hat{u}}: D(\lambda_{\mathcal{F}}^*/2) \rightarrow \tilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2)$. So for $\xi = (\psi_{x, \hat{u}}^g)^{-1}(y)$, then $\tilde{y} = \widetilde{\exp}_{\tilde{x}, \hat{u}}(\xi)$ by definition.

Similarly, \tilde{d}' denotes the distance function induced on $\tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2)$, and \tilde{g}' denotes the induced metric tensor on $\tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2)$. The geodesic coordinates associated to \tilde{g}' and \hat{v}' , centered at \tilde{y}' , are denoted by $\widetilde{\exp}_{\tilde{y}', \hat{v}'}: D(\lambda_{\mathcal{F}}^*/2) \rightarrow \tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2)$. Then the map $\Psi_{x, y'}$ from (122) can be expressed by

$$\Psi_{x, y'} = \widetilde{\exp}_{\tilde{y}', \hat{v}'}^{-1} \circ \widetilde{\exp}_{\tilde{x}, \hat{u}} \circ T_{\xi}$$

and there is the diagram of maps:

$$\begin{array}{ccccc} T_x \mathcal{F} \cong \mathbb{R}^n \supset D(\lambda_{\mathcal{F}}^*/2) & \xrightarrow{\widetilde{\exp}_{\tilde{x}, \hat{u}} \circ T_{\xi}} & \tilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2) \subset (-1, 1)^n \times \{w_x\} & \xrightarrow{\varphi_i^{-1}} & \mathcal{P}_i(w_x) \\ (123) \quad \Psi_{x, y'} \downarrow & & = \downarrow & & \downarrow \phi_i(x, x') \\ T_{y'} \mathcal{F} \cong \mathbb{R}^n \supset D(\lambda_{\mathcal{F}}^*/2) & \xrightarrow{\widetilde{\exp}_{\tilde{y}', \hat{v}'}} & \tilde{D}'_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2) \subset (-1, 1)^n \times \{w_{y'}\} & \xrightarrow{\varphi_i^{-1}} & \mathcal{P}_i(w_{y'}) \end{array}$$

Set $\vec{\gamma} = \tilde{x} + \hat{u} \cdot \xi$ so that $T_{\vec{\gamma}} \circ F_{\hat{u}}(\vec{a}) = T_{\tilde{x}} \circ F_{\hat{u}}(\vec{a} + \xi)$.

By condition (102) of Definition 13.1, and using that $\lambda_{\mathcal{F}}^* \leq \lambda_{\varepsilon_0}$, for all $\vec{a} \in D(\lambda_{\mathcal{F}}^*/2)$ we have

$$(124) \quad \tilde{d}(\widetilde{\exp}_{\tilde{x}, \hat{u}}(\vec{a} + \xi), T_{\vec{\gamma}} \circ F_{\hat{u}}(\vec{a})) \leq \varepsilon_0 \lambda_{\mathcal{F}}^*, \quad \tilde{d}'(\widetilde{\exp}_{\tilde{y}', \hat{v}'}(\vec{a}), T_{\tilde{y}'} \circ F_{\hat{v}'}(\vec{a})) \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$$

Set $\vec{a} = \vec{0}$ in the first estimate of (124), then by Lemma 13.2 we obtain

$$(125) \quad \|\tilde{y} - \vec{\gamma}\|_{\hat{u}} \leq \tilde{d}(\tilde{y}, \vec{\gamma}) + \varepsilon_0 \lambda_{\mathcal{F}}^* = \tilde{d}(\widetilde{\exp}_{\tilde{x}, \hat{u}}(\xi), T_{\vec{\gamma}}(\vec{0})) + \varepsilon_0 \lambda_{\mathcal{F}}^* \leq 2\varepsilon_0 \lambda_{\mathcal{F}}^*$$

The “obvious” next step is to replace the orthonormal framing \hat{v}' for \mathbb{R}^n for the norm $\|\cdot\|_{\hat{v}'}$ with the new framing $\hat{v} = \hat{u}$, and then the claim of Proposition 15.4 would follow. However, \hat{u} need not

be an orthonormal framing the norm $\|\cdot\|_{\hat{v}'}$, so it is necessary to adjust the framing \hat{u} using the Gram-Schmidt orthogonalization process. This introduces additional errors, which depend on the “distance” from \hat{u} to \hat{v}' in the Lie group $GL(\mathbb{R}^n)$. We formulate this error as follows, using estimates derived from the Gram-Schmidt orthogonalization process. This derivation of the following result is straightforward, and we omit the proof.

LEMMA 15.5. *Let \mathbb{R}^n have the standard Euclidean inner product with norm $\|\cdot\|$. There exists $\epsilon_n > 0$ and a monotone continuous function $\delta_n: [0, \epsilon_n] \rightarrow [0, \epsilon_n]$ with $\delta_n(0) = 0$, such that given $0 < \epsilon \leq \epsilon_n$ set $\delta = \delta_n(\epsilon) > 0$, then for any basis $\{\vec{f}_1, \dots, \vec{f}_n\} \subset \mathbb{R}^n$, whose vectors satisfy*

- (1) $1 - \delta < \|\vec{f}_j'\| < 1 + \delta$, for $1 \leq j \leq n$,
- (2) $|\vec{f}_i' \bullet \vec{f}_j'| < \delta$, for $1 \leq i \neq j \leq n$,

then there exists orthonormal vectors $\{\vec{f}_1, \dots, \vec{f}_n\}$ such that $\|\vec{f}_j - \vec{f}_j'\| \leq \epsilon$. \square

Recall that $\hat{e} = \{\vec{e}_1, \dots, \vec{e}_n\}$ is the standard orthogonal basis for \mathbb{R}^n . Scale these unit vectors by a factor of $d_2 = \lambda_{\mathcal{F}}^*/5$ so they lie in the domain of $\Psi_{x,y'}$, and set $\vec{z}_j = F_{\hat{u}}(d_2 \vec{e}_j)$. Note that $\|\vec{z}_j\|_{\hat{u}} = d_2$.

Recall that $\xi = (\psi_{x,\hat{u}}^g)^{-1}(y)$. Then $d_2 \vec{e}_j + \xi \in D(\tilde{x}, \lambda_{\mathcal{F}}^*/2)$ so that $\tilde{z}_j = \widetilde{\exp}_{\tilde{x}, \hat{u}}(d_2 \vec{e}_j + \xi) \in \tilde{D}_i(\tilde{x}, \lambda_{\mathcal{F}}^*/2)$ is well-defined. Then by Lemma 15.2,

$$(126) \quad |\tilde{d}(\tilde{z}_j, \tilde{y}) - d_2| = |\tilde{d}(\tilde{z}_j, \tilde{y}) - \|\tilde{z}_j\|_{\hat{u}}| = |\tilde{d}(\tilde{z}_j, \tilde{y}) - \|\hat{u} \cdot (d_2 \vec{e}_j + \xi) - \hat{u} \cdot \xi\|_{\hat{u}}| \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$$

$$(127) \quad |\tilde{d}(\tilde{z}_j, \tilde{z}_k) - \sqrt{2}d_2| = |\tilde{d}(\tilde{z}_j, \tilde{z}_k) - \|\hat{u} \cdot (d_2 \vec{e}_j + \xi) - \hat{u} \cdot (d_2 \vec{e}_k + \xi)\|_{\hat{u}}| \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$$

The estimates (126) and (127) imply that the set $\{(\tilde{z}_1 - \tilde{y})/d_2, \dots, (\tilde{z}_n - \tilde{y})/d_2\}$ is an “almost orthonormal” collection for the metric \tilde{d} .

By Lemma 15.3, the map $\phi_i(x, x')$ is an $\varepsilon_0 \lambda_{\mathcal{F}}^*$ -isometry, and as $\phi_i(x, x')$ is the identity map in the coordinates φ_i , we obtain estimates corresponding to (126) and (127) for the metric \tilde{d}' ,

$$(128) \quad |\tilde{d}'(\tilde{z}_j, \tilde{y}) - d_2| \leq 2\varepsilon_0 \lambda_{\mathcal{F}}^* \implies \|\tilde{z}_j - \tilde{y}\|_{\hat{v}'} - d_2 \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

$$(129) \quad |\tilde{d}'(\tilde{z}_j, \tilde{z}_k) - \sqrt{2}d_2| \leq 2\varepsilon_0 \lambda_{\mathcal{F}}^* \implies \|\tilde{z}_j - \tilde{z}_k\|_{\hat{v}'} - \sqrt{2}d_2 \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

which follow from estimate (106) of Lemma 13.2.

Define $\vec{z}_j' = \widetilde{\exp}_{\tilde{y}', \hat{v}'}^{-1}(\tilde{z}_j)$. Then by estimate (107) of Lemma 13.2, and noting that $\vec{z}_j' \in \tilde{D}_i(\tilde{y}', \lambda_{\mathcal{F}}^*/2)$,

$$(130) \quad \|\tilde{z}_j - T_{\tilde{y}'} \circ F_{\hat{v}'}(\vec{z}_j')\|_{\hat{v}'} = \|\widetilde{\exp}_{\tilde{y}', \hat{v}'}(\vec{z}_j') - T_{\tilde{y}'} \circ F_{\hat{v}'}(\vec{z}_j')\|_{\hat{v}'} \leq \varepsilon_0 \lambda_{\mathcal{F}}^*$$

Then by (128) and (129), and using that the map $T_{\tilde{y}'} \circ F_{\hat{v}'}$ is an isometry from the norm $\|\cdot\|$ to the norm $\|\cdot\|_{\hat{v}'}$, we obtain for the Euclidean norm on \mathbb{R}^n , for $1 \leq j \neq k \leq n$,

$$(131) \quad \|\vec{z}_j' - d_2\| \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

$$(132) \quad \|\vec{z}_j' - \vec{z}_k'\| - \sqrt{2}d_2 \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

Set $\vec{f}_j' = \vec{z}_j'/d_2$, and observe that (131) implies the collection $\{\vec{f}_1', \dots, \vec{f}_n'\}$ satisfies hypothesis (1) of Lemma 15.5 for $\delta = 15\varepsilon_0$.

It remains to estimate $|\vec{f}_j' \bullet \vec{f}_k'|$ for $1 \leq j \neq k \leq n$. Write $\vec{f}_k' = \vec{f}_{j,k}' + \vec{f}_{k,k}'$ where $\vec{f}_{j,k}'$ is collinear with \vec{f}_j' and $\vec{f}_j' \bullet \vec{f}_{k,k}' = 0$. Then $|\vec{f}_j' \bullet \vec{f}_k'| = |\vec{f}_j' \bullet \vec{f}_{j,k}'| = \|\vec{f}_j'\| \|\vec{f}_{j,k}'\|$.

Note also that $\|\vec{f}_j'\|^2 = \|\vec{f}_{k,k}'\|^2 + \|\vec{f}_{j,k}'\|^2$ hence $\|\vec{f}_{k,k}'\|^2 = (\|\vec{f}_j'\|^2 - \|\vec{f}_{j,k}'\|^2)$. By (132) we have

$$(133) \quad \sqrt{2} - 15\varepsilon_0 \leq \|\vec{f}_j' - \vec{f}_k'\| = \|(\vec{f}_j' - \vec{f}_{j,k}') - \vec{f}_{k,k}'\| \leq \sqrt{2} + 15\varepsilon_0$$

After squaring and using the orthogonality of the vectors, we obtain

$$2 - 30\sqrt{2}\varepsilon_0 + 225\varepsilon_0^2 \leq \|(\vec{f}_j' - \vec{f}_{j,k}')\|^2 + \|\vec{f}_{k,k}'\|^2 \leq 2 + 30\sqrt{2}\varepsilon_0 + 225\varepsilon_0^2$$

Note that \vec{f}_j and $\vec{f}_{j,k}$ are collinear, hence $\|\vec{f}_j - \vec{f}_{j,k}\|^2 = \|\vec{f}_j\|^2 - 2\|\vec{f}_j\| \cdot \|\vec{f}_{j,k}\| + \|\vec{f}_{j,k}\|^2$. Then using that $2 - 100\varepsilon_0 < 2 - 30\sqrt{2}\varepsilon_0 + 225\varepsilon_0^2$, we have

$$2 - 100\varepsilon_0 < \|\vec{f}_j\|^2 - 2\|\vec{f}_j\| \cdot \|\vec{f}_{j,k}\| + \|\vec{f}_{j,k}\|^2 + \|\vec{f}_{k,k}\|^2 < 2 + 100\varepsilon_0$$

From the identity $\|\vec{f}_{k,k}\|^2 = (\|\vec{f}_j\|^2 - \|\vec{f}_{j,k}\|^2)$ one derives $\|\vec{f}_j \bullet \vec{f}_k\| = \|\vec{f}_{j,k}\| < 100\varepsilon_0$.

Thus, the collection $\{\vec{f}_1, \dots, \vec{f}_n\}$ satisfies both hypotheses of Lemma 15.5 for $\delta = 100\varepsilon_0$. We assume that $\varepsilon_0 < \varepsilon_4/20$ in (116) so that $\varepsilon_4 < \varepsilon_5 = 2\varepsilon_4 - 20\varepsilon_0$. By choice of ε_0 in (116), we have $100\varepsilon_0 < \delta_n(\varepsilon_5)$ so we obtain the orthonormal framing $\hat{f} = \{\vec{f}_1, \dots, \vec{f}_n\}$ of \mathbb{R}^n satisfying $\|\vec{f}_k - \vec{f}'_k\| \leq \varepsilon_5$

Define $\vec{v}_j = F_{\hat{v}'}(\vec{f}_j)$, then $\hat{v} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal frame for the norm $\|\cdot\|_{\hat{v}'}$ so defines an orthonormal framing of $T_{y'}\mathcal{F}$. Note that $F_{\hat{v}} = F_{\hat{v}'} \circ F_{\hat{f}}$ and calculate:

$$\begin{aligned} & \|T_{\vec{\gamma}} \circ F_{\hat{u}}(d_2 \vec{e}_j) - T_{\vec{y}'} \circ F_{\hat{v}}(d_2 \vec{e}_j)\|_{\hat{v}'} \\ & \leq \| (T_{\vec{\gamma}} \circ F_{\hat{u}}(d_2 \vec{e}_j) - \tilde{z}_j) \|_{\hat{v}'} + \|\tilde{z}_j - T_{\vec{y}'} \circ F_{\hat{v}}(d_2 \vec{e}_j)\|_{\hat{v}'} \\ & = \|T_{\vec{\gamma}} \circ F_{\hat{u}}(d_2 \vec{e}_j) - \tilde{z}_j\|_{\hat{v}'} + \|\tilde{z}_j - T_{\vec{y}'} \circ F_{\hat{v}'}(d_2 \vec{f}_j)\|_{\hat{v}'} \\ & \leq \|T_{\vec{\gamma}} \circ F_{\hat{u}}(d_2 \vec{e}_j) - \tilde{z}_j\|_{\hat{v}'} + \|\tilde{z}_j - T_{\vec{y}'} \circ F_{\hat{v}'}(d_2 \vec{f}_j)\|_{\hat{v}'} + \|T_{\vec{y}'} \circ F_{\hat{v}'}(d_2 \vec{f}_j) - T_{\vec{y}'} \circ F_{\hat{v}'}(d_2 \vec{f}_j)\|_{\hat{v}'} \\ & \leq 3\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_0 \lambda_{\mathcal{F}}^* + 2d_2 \varepsilon_5 = 4\varepsilon_0 \lambda_{\mathcal{F}}^* + 2\varepsilon_5 \lambda_{\mathcal{F}}^*/5 \end{aligned}$$

where we use successively the definitions of the quantities involved, Lemmas 13.2 and 15.2, the estimate (124), the estimate (130), and Lemma 15.5. Then by the approximations (124) and Lemmas 13.2 and 15.2, we have for all $\vec{a} \in D(2\lambda_{\mathcal{F}}^*/5)$ that

$$(134) \quad \tilde{d}(\widetilde{\exp}_{\vec{x}, \vec{u}}(\vec{a} + \xi), \widetilde{\exp}_{\vec{y}', \hat{v}}(\vec{a})) \leq 7\varepsilon_0 \lambda_{\mathcal{F}}^* + 2\varepsilon_5 \lambda_{\mathcal{F}}^*/5$$

Hence by Lemma 15.2 and our choice $\varepsilon_5 = 2\varepsilon_4 - 20\varepsilon_0$, we obtain

$$(135) \quad \|\Psi_{x, y'}(\vec{a}) - \vec{a}\| = \|\widetilde{\exp}_{\vec{y}', \hat{v}}^{-1} \circ \widetilde{\exp}_{\vec{x}, \vec{u}} \circ T_{\xi}(\vec{a})\| \leq 8\varepsilon_0 \lambda_{\mathcal{F}}^* + 2\varepsilon_5 \lambda_{\mathcal{F}}^*/5 < \varepsilon_4 \lambda_{\mathcal{F}}^*$$

completing the proof of Proposition 15.4. \square

15.2. Robustness criteria. The fine control of the affine structure of geodesic coordinates provided by Proposition 15.4 is used in establishing robustness criteria for leafwise Delaunay triangulations in the next section. In preparation, we define a “non-linear” form of the robustness criteria in Definition 11.6 and Proposition 12.3.

Recall that $\text{Span}(\vec{v}_0, \dots, \vec{v}_k) \subset \mathbb{R}^n$ is the *affine* span of the vectors $\{\vec{v}_0, \dots, \vec{v}_k\}$.

DEFINITION 15.6. *Let $\rho > 0$ and $x \in U_i$ such that $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*) \subset \mathcal{P}_i(x)$. Let $1 \leq m \leq n$. A set $\{y_0, \dots, y_m\} \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)$ is ρ -robust if for each $1 \leq k < m$, the following leafwise metric conditions hold:*

- (1) Fix an orthonormal frame $\hat{u} = \{\vec{u}_1, \dots, \vec{u}_n\} \subset T_{y_k}\mathcal{F}$ with geodesic coordinates $\psi_{y_k, \hat{u}}^g$;
- (2) for each $0 \leq j \leq k$, set $\vec{v}_j = (\psi_{y_k, \hat{u}}^g)^{-1}(y_j)$;
- (3) Set $H(y_0, \dots, y_k; y_k) = \psi_{y_k, \hat{u}}^g \{\text{Span}(\vec{v}_0, \dots, \vec{v}_k) \cap D(\lambda_{\mathcal{F}}^*)\}$.

Then the point y_{k+1} lies at distance at least ρ from the submanifold $H(y_0, \dots, y_k; y_k)$

We show that the robustness condition for points in Definition 15.6 implies the robustness condition Definition 11.6 holds for their vector coordinates in geodesic normal coordinates.

PROPOSITION 15.7. *Given constants*

$$(136) \quad \lambda_{\mathcal{F}}^*/10 = d_1 < d_1 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* < e_1 < e_2 < d_2 - 2\varepsilon_0 \lambda_{\mathcal{F}}^* < d_2 = 2\lambda_{\mathcal{F}}^*/10$$

and $0 < \rho_0 < d_1$, let $x \in \mathfrak{M}$ and suppose $\{y_0, \dots, y_m\} \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*/2)$ satisfy

- (1) $e_1 \leq d_{\mathcal{F}}(y_j, y_k) \leq 2e_2$ for $0 \leq j \neq k \leq m$
- (2) $\{y_0, \dots, y_m\}$ is ρ_0 -robust.

Given an orthonormal frame \hat{u} of $T_x\mathcal{F}$, set $\vec{w}_j = (\psi_{x,\hat{u}}^g)^{-1}(y_j) \in D(\lambda_{\mathcal{F}}^*/2)$ for $0 \leq j \leq m$. Then $\{\vec{w}_0, \dots, \vec{w}_m\} \subset \mathbb{R}^n$ is ρ_m -robust, where $\rho_\ell = \rho_\ell(\rho_0, \varepsilon_4\lambda_{\mathcal{F}}^*, d_1, d_2)$ is defined by (112) for $1 \leq \ell \leq m$.

Proof. We proceed by induction. By assumption, $d_{\mathcal{F}}(y_0, y_1) \geq e_1 > d_1 + \varepsilon_0\lambda_{\mathcal{F}}^*$, so by Lemma 15.2 we have $\|\vec{w}_1 - \vec{w}_0\| \geq d_1 \geq \rho_0 - 2\varepsilon_4\lambda_{\mathcal{F}}^* = \rho_1$.

Now assume that the collection $\{\vec{w}_0, \dots, \vec{w}_\ell\}$ is ρ_ℓ -robust, for each $1 \leq \ell < m$. We show that $\{\vec{w}_0, \dots, \vec{w}_{\ell+1}\}$ is $\rho_{\ell+1}$ -robust.

Let U_i be a coordinate chart such that $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}^*) \subset U_i$.

Let $T_{\vec{w}_\ell}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T_{\vec{w}_\ell}(\vec{x}) = \vec{x} + \vec{w}_\ell$ be translation by \vec{w}_ℓ . Define the composition

$$\Psi_\ell \equiv (\psi_{y_\ell, \hat{v}}^g)^{-1} \circ \psi_{x, \hat{u}}^g \circ T_{\vec{w}_\ell}: D(2d_2) \rightarrow D(\lambda_{\mathcal{F}}^*/2)$$

for an orthonormal frame $\hat{v} = \hat{v}_\ell$ of $T_{y_\ell}\mathcal{F}$ as provided by Proposition 15.4 so that $\|\Psi_\ell(\vec{x}) - \vec{x}\| \leq \varepsilon_4\lambda_{\mathcal{F}}^*$, where ε_4 is defined by (114) and (115). This is possible by our choice of ε_0 in (116). (In this application of Proposition 15.4, we take $y' = y_\ell \in \mathcal{P}_i(x)$ which is on the same plaque as x .)

For $0 \leq j \leq m$, define $\vec{z}_j = (\psi_{y_\ell, \hat{v}}^g)^{-1}(y_j)$, and also set $\vec{w}'_j = \vec{w}_j - \vec{w}_\ell$. Then $\Psi_\ell(\vec{w}'_j) = \vec{z}_j$. Using that Ψ_ℓ is $\varepsilon_4\lambda_{\mathcal{F}}^*$ close to the identity, we have that each $\|\vec{z}_j - \vec{w}'_j\| \leq \varepsilon_4\lambda_{\mathcal{F}}^*$.

Note that $\{\vec{w}_0, \dots, \vec{w}_\ell\}$ is ρ_ℓ -robust if and only if the collection $\{\vec{w}'_0, \dots, \vec{w}'_\ell\}$ is ρ_ℓ -robust.

By the definition that $\{y_0, \dots, y_m\}$ is ρ_0 -robust, the point $y_{\ell+1}$ lies at distance at least ρ_0 from the submanifold $\psi_{y_\ell, \hat{v}}^g(\text{Span}(\vec{z}_0, \dots, \vec{z}_\ell))$. So by Lemma 15.2, the vector $\vec{z}_{\ell+1}$ lies at distance at least $\rho_0 - \varepsilon_0\lambda_{\mathcal{F}}^* \geq \rho_\ell$ from the linear span $\text{Span}(\vec{z}_0, \dots, \vec{z}_\ell)$. Thus, we also have that the collection $\{\vec{z}_0, \dots, \vec{z}_{\ell+1}\}$ is ρ_ℓ -robust.

It is given that $e_1 \leq d_{\mathcal{F}}(y_j, y_k) \leq e_2$ for $0 \leq j \neq k \leq m$, so by Lemma 15.2 we have

$$d_1 < e_1 - 2\varepsilon_0\lambda_{\mathcal{F}}^* \leq \|\vec{z}_j - \vec{z}_k\| \leq 2e_2 + 2\varepsilon_0\lambda_{\mathcal{F}}^* < 2d_2$$

for all $0 \leq j \neq k \leq m$. We can thus apply Proposition 12.3 for $\varepsilon = \varepsilon_4\lambda_{\mathcal{F}}^*$ to the collection $\{\vec{z}_0, \dots, \vec{z}_{\ell+1}\}$ to conclude that $\text{Span}(\vec{w}_0, \dots, \vec{w}_{\ell+1})$ is $\rho_{\ell+1}$ -robust. \square

PART V - NICE STABLE TRANSVERSALS

Given a proper base \tilde{K}_0 along with a \tilde{K}_0 -admissible transverse set V_0 which satisfies prescribed metric restrictions, we give the construction of a nice stable transversal \mathcal{X} for an open neighborhood of \tilde{K}_0 . The procedure is inductive, in that we assume that a collection of transversals are given, which satisfy a set of regularity conditions, and then prove that it is possible to extend the collection by adding another transversal so the regularity conditions are again satisfied. This process continues until we obtain a (d_1, d_2) -uniform transversal for the Reeb neighborhood $\mathfrak{N}(\tilde{K}_0, V_0)$ defined by (21). This will complete the proof of Theorem 10.1.

16. REGULAR PARTIAL TRANSVERSALS

We recall the given data, and fix notations convenient for the proofs below. Let $L_0 \subset \mathfrak{M}$ be a leaf without holonomy. Since the holonomy covering map $\Pi: \tilde{L}_0 \rightarrow L_0$ is a diffeomorphism, we work with points and sets in \mathfrak{M} instead of the holonomy covering, unless it is necessary to emphasize that the construction is taking place in a subset of the foliated microbundle \mathfrak{N}_0 over \tilde{L}_0 . This simplifies the notation throughout.

Let \mathcal{M}_0 be an (e_1, e_2) -net for L_0 as in Section 5.1, where $e_2 = \epsilon_{\mathcal{U}}^{\mathcal{F}}/4$. There is a corresponding net $\tilde{\mathcal{M}}_0 = \Pi^{-1}(\mathcal{M}_0)$ in \tilde{L}_0 , where $\tilde{z} \in \tilde{\mathcal{M}}_0$ corresponds to $z = \Pi(\tilde{z}) \in \mathcal{M}_0$.

Assume there is given the covering of \mathfrak{M} by coordinate charts $\{U_{i_z} \mid z \in \mathcal{M}_0\}$, as in the proof of Lemma 5.2, where $1 \leq i_z \leq \nu$, and such that $B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$. Correspondingly, for each $\tilde{z} \in \tilde{\mathcal{M}}_0$, there is a foliation chart $\tilde{U}_{\tilde{z}} = \overline{U}_{i_z} \times \{\tilde{z}\}$ for \mathfrak{N}_0 as discussed in Section 5.2.

Let $K_0 \subset L_0$ be a connected compact subset which is a union of the plaques in L_0 . We may assume there is a subset $\mathcal{M}'_{K_0} \subset \mathcal{M}_{K_0} = \mathcal{M}_{K_0}$ so that $K_0 = \cup \{\mathcal{P}_{i_z}(z) \mid z \in \mathcal{M}'_{K_0}\}$. Set $\tilde{K}_0 = \Pi^{-1}(K_0)$ and $\tilde{\mathcal{M}}_{K_0} = \Pi^{-1}(\mathcal{M}_{K_0}) \subset \tilde{L}_0$.

Let R_0 denote the diameter of K_0 in the leafwise metric, so for any $x \in K_0$ we have $K_0 \subset D_{\mathcal{F}}(x, R_0)$.

Fix a basepoint $z_0 \in \mathcal{M}_{K_0}$. Without loss of generality, we may assume that $i_{z_0} = 1$, and let $w_0 \in \mathfrak{T}_1$ be the projection of z_0 to the transverse space for U_1 .

For $z \in \mathcal{M}_0$ let h_z denote the holonomy along some nice path γ from z_0 to z . This is well defined as L_0 is without holonomy.

Recall that the constant $\epsilon_{\mathcal{U}}^{\mathcal{F}}$ is the “leafwise Lebesgue number” defined in equation (6), so that for all $y \in \mathfrak{M}$, $D_{\mathcal{F}}(y, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset D_{\mathfrak{M}}(y, \epsilon_{\mathcal{U}}/2)$. The constants $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, and $\lambda_{\mathcal{F}}^*$ are as chosen in Section 14. Also, r_* was determined by the choice of $\lambda_{\mathcal{F}}^*$ in Section 14.5.

For $r = R_0 + \delta_{\mathcal{U}}^{\mathcal{F}}$, let $\delta_0^{\mathcal{T}} = \delta(r_*/2, r) \leq \epsilon$ be the constant defined in Proposition 4.9, so that $D_{\mathfrak{X}}(w_0, \delta_0^{\mathcal{T}}) \subset D(h_z)$ for all $z \in \mathcal{M}_{K_0}$. Moreover, $h_z(D_{\mathfrak{X}}(w_0, \delta_0^{\mathcal{T}})) \subset D_{\mathfrak{X}}(h_z(w_0), \epsilon)$.

Let $V_0 \subset D_{\mathfrak{X}}(w_0, \delta_0^{\mathcal{T}})$ be a clopen subset with $w_0 \in V_0$. Then V_0 is K_0 -admissible, so we can form the Reeb neighborhood $\mathfrak{N}(\tilde{K}_0, V_0) \subset \tilde{\mathfrak{N}}_0$ as in Definition 5.8. Note that even though the map $\tilde{L}_0 \rightarrow L_0 \subset \mathfrak{M}$ is assumed to be injective, the same is not necessarily true for the map $\mathfrak{N}(\tilde{K}_0, V_0) \rightarrow \mathfrak{M}$. For this reason, constructions using subsets of the coordinate neighborhoods are formed in the space $\tilde{\mathfrak{N}}_0$. Notations involving the “tilde” indicate that the construction is considered in $\tilde{\mathfrak{N}}_0$ as opposed to \mathfrak{M} .

For all $z \in \mathcal{M}_{K_0}$, set $V_z = h_z(V_0) \subset \mathfrak{T}_{i_z}$ and $w_z = h_z(w_0)$. Then by the definition of the function $\delta(\epsilon, r)$ and the choice of $V_0 \subset D_{\mathfrak{X}}(w_0, \delta_0^{\mathcal{T}})$, we have

$$V_z \subset B_{\mathfrak{X}}(w_z, r_*/2) \subset \mathfrak{T}_{i_z}$$

Hence, for any $w \in V_z$ we have $V_z \subset B_{\mathfrak{X}}(w, r_*)$. Also, define the compact sets

$$(137) \quad \mathfrak{U}_z^V = \pi_z^{-1}(V_z) \subset U_{i_z} \quad , \quad \tilde{\mathfrak{U}}_z^V = \mathfrak{U}_z^V \times \{\tilde{z}\} \subset \tilde{U}_{i_z} \subset \tilde{\mathfrak{N}}_0$$

For $x \in \mathfrak{U}_z^V$, define a *standard sections* by

$$(138) \quad \mathfrak{Z}(x, i_z, V_z) = \varphi_{i_z}^{-1}(\lambda_{i_z}(x), V_z) \subset \mathfrak{U}_z^V \quad , \quad \tilde{\mathfrak{Z}}(x, i_z, V_z) \equiv \mathfrak{Z}(x, i_z, V_z) \times \{\tilde{z}\} \subset \tilde{\mathfrak{N}}_0$$

The construction of a nice stable transversal \mathcal{X} proceeds by induction from a given “partial net” in K_0 by choosing points which complete it to a (d_1, d_2) -uniform net. The net points $\xi_k \in K_0$ chosen must satisfy appropriate net and general position conditions. Ultimately, we let \mathcal{X} be the union of the standard transversals $\mathcal{X}_k = \mathfrak{Z}(\xi_k, i_{\xi_k}, V_0)$ through these points. The procedure for making these choices utilizes the constants and estimates from previous sections. In addition, introduce a sequence of constants based on the scale $\lambda_{\mathcal{F}}^*$:

$$(139) \quad \begin{aligned} d_1 &= .10 \cdot \lambda_{\mathcal{F}}^*, & d'_1 &= .11 \cdot \lambda_{\mathcal{F}}^*, & d''_1 &= .12 \cdot \lambda_{\mathcal{F}}^* \\ d_2 &= .20 \cdot \lambda_{\mathcal{F}}^*, & d'_2 &= .19 \cdot \lambda_{\mathcal{F}}^*, & d''_2 &= .18 \cdot \lambda_{\mathcal{F}}^* \end{aligned}$$

Note that $d_2 = 2d_1$ and

$$\lambda_{\mathcal{F}}^*/10 = d_1 < d'_1 < d''_1 < d''_2 < d'_2 < d_2 = \lambda_{\mathcal{F}}^*/5$$

DEFINITION 16.1. *Let $V_0 \subset D_{\mathfrak{X}}(w_0, \delta_0^{\mathcal{T}})$ be a clopen subset which is K_0 -admissible, and assume that the leaf L_0 determined by $w_0 \in V_0$ has no germinal holonomy. Then $\hat{\mathcal{X}}_p = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_p$ is a regular partial V_0 -transversal for K_0 if there are given:*

- (i) points $\Xi_p = \{\xi_1, \dots, \xi_p\} \subset K_0$;
- (ii) points $\Lambda_p = \{z_1, \dots, z_p\} \subset \mathcal{M}_{K_0}$ such that $\xi_j \in B_{\mathcal{F}}(z_j, \epsilon_{\mathcal{U}}^{\mathcal{F}}/2)$;
- (iii) indices $\theta_j = i_{z_j}$ for $1 \leq j \leq p$ for $z_{\theta_j} \in \mathcal{M}_{K_0}$

such that $\mathcal{X}_j = \mathfrak{Z}(\xi_j, \theta_j, V_0)$ for $1 \leq j \leq p$. That is, \mathcal{X}_j is a standard section through ξ_j .

Moreover, for $x \in \mathfrak{M}$ define $\mathcal{N}_p(x) = \widehat{\mathcal{X}}_p \cap L_x$. Then we require

(iv) for all $y \neq z \in \mathcal{N}_p(x)$, then $d_{\mathcal{F}}(y, z) \geq d'_1$.

Define $\mathcal{N}_p = \widehat{\mathcal{X}}_p \cap L_0$ for the case where $x \in L_0$.

Note that if $p = 1$ in Definition 16.1, then condition (iv) is always satisfied: given $\xi \neq \xi' \in \mathcal{N}_1$, then $\xi, \xi' \in \mathcal{X}_1$ implies they are contained in distinct plaques of L_0 , hence $d_{\mathcal{F}}(\xi, \xi') \geq 2\delta_{\mathcal{U}}^{\mathcal{F}}$.

For $p \geq 2$, we impose two additional stability conditions criteria on the nets $\mathcal{N}_p(x)$.

Let $\Delta'_{\mathcal{F}}(\widehat{\mathcal{X}}_p)$ denote the leafwise simplicial complex for $\widehat{\mathcal{X}}_p$ defined using the constants (d'_1, d'_2) in (139) above. That is, a $(k+1)$ -tuple $\{y_0, \dots, y_k\} \subset \widehat{\mathcal{X}}_p$ defines a k -simplex $\Delta(y_0, \dots, y_k) \in \Delta'_{\mathcal{F}}(\widehat{\mathcal{X}}_p)$ if there exists $\omega \in L_{y_0}$ and $0 < r \leq d'_2$ such that $B_{\mathcal{F}}(\omega, r) \cap \widehat{\mathcal{X}}_p = \emptyset$ and $\{y_0, \dots, y_k\} \subset S_{\mathcal{F}}(\omega, r) \cap \widehat{\mathcal{X}}_p$.

Let $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\widehat{\mathcal{X}}_p)$ be an n -simplex, then by Proposition 8.6, the collection of hyperplanes $\{L(y_0, y_i) \mid 1 \leq i \leq n\}$ in $D_{\mathcal{F}}(y_0, \lambda_{\mathcal{F}})$ have non-trivial intersection:

$$(140) \quad \omega(y_0, \dots, y_n) = L(y_0, y_1) \cap \dots \cap L(y_0, y_n) \cap \mathcal{C}(y_0)$$

where $\mathcal{C}(y_0)$ is the cell determined by y_0 in the Voronoi tessellation of K_0 associated to the net $\mathcal{N}_p(y_0)$. The point $\omega(y_0, \dots, y_n) \in L_{y_0}$ is the center of the associated inscribed closed disk with radius $r(y_0, \dots, y_n) = d_{\mathcal{F}}(y_\ell, \omega(y_0, \dots, y_n))$ for all $0 \leq \ell \leq n$.

DEFINITION 16.2. The transversal $\widehat{\mathcal{X}}_p$ for K_0 is ε_1 -regular if for all $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\widehat{\mathcal{X}}_p)$, and for all $\xi \in \mathcal{N}_p(y_0) - \{y_0, \dots, y_n\}$,

$$(141) \quad d_{\mathcal{F}}(\xi, \omega(y_0, \dots, y_n)) \geq r(y_0, \dots, y_n) + \varepsilon_1 \lambda_{\mathcal{F}}^*$$

Given a simplex $\Delta(y_0, \dots, y_k) \in \Delta'_{\mathcal{F}}(\widehat{\mathcal{X}}_p)$, we say that the vertices are *properly ordered* if there exists $1 \leq i_0 < i_1 < \dots < i_k \leq p$ and points $y_{i_j} \in \widehat{\mathcal{X}}_p \subset L_{y_0}$ such that $y_j = \mathcal{X}_{i_j} \cap \mathcal{P}_{\theta_{i_k}}(y_k)$.

DEFINITION 16.3. Let $\rho = 3\varepsilon_2 \lambda_{\mathcal{F}}^*/2$. A regular partial V_0 -transversal $\widehat{\mathcal{X}}_p$ for K_0 is ρ -robust if for all $\Delta(x_0, \dots, x_n) \in \Delta'_{\mathcal{F}}(\widehat{\mathcal{X}}_p)$, the collection $\{x_0, \dots, x_n\}$ is ρ -robust, in the sense of Definition 15.6, where we assume the vertices $\{x_0, \dots, x_n\}$ are properly ordered.

17. CONSTRUCTION OF A COMPLETE REGULAR V_0 -TRANSVERSAL

In the following, we assume there is given a regular partial V_0 -transversal $\widehat{\mathcal{X}}_p$ for K_0 with $p \geq 1$, which satisfies the conditions of Definitions 16.2 and 16.3. Without loss of generality, we also assume that \mathcal{X}_1 is defined by

$$(142) \quad \xi_1 = z_0 \in K_0, \quad \Xi_1 = \{\xi_1\}, \quad \Lambda_1 = \{z_1\}, \quad \theta_1 = i_{z_1} = 1, \quad \mathcal{X}_1 = \mathfrak{Z}(\xi_1, \theta_1, V_0)$$

DEFINITION 17.1. A regular partial V_0 -transversal $\widehat{\mathcal{X}}_p$ for K_0 is δ -complete if

$$K_0 \subset \text{Pen}_{\mathcal{F}}(\mathcal{N}_p, \delta) = \bigcup_{z \in \mathcal{N}_p} D_{\mathcal{F}}(z, \delta)$$

We formulate the inductive construction for transversals.

PROPOSITION 17.2. Let $\widehat{\mathcal{X}}_p$ for $p \geq 1$ be a regular partial V_0 -transversal satisfies the conditions of Definitions 16.2 and 16.3. If $\widehat{\mathcal{X}}_p$ is not d'_2 -complete for K_0 then there exists $\xi_{p+1} \in K_0$ so that for

- $\Xi_{p+1} = \{\xi_1, \dots, \xi_{p+1}\} \subset L_0$
- $\Lambda_{p+1} = \{z_1, \dots, z_p, z_{p+1}\} \subset \mathcal{M}_{K_0}$ such that $\xi_j \in B_{\mathcal{F}}(z_j, \epsilon_{\mathcal{U}}^{\mathcal{F}}/2)$
- $\theta_j = i_{z_j}$ for $1 \leq j \leq p+1$ with $z_{\theta_j} \in \mathcal{M}_{K_0}$
- $\mathcal{X}_{p+1} = \mathfrak{Z}(\xi_{p+1}, \theta_{p+1}, V_0)$, $\widehat{\mathcal{X}}_{p+1} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_p \cup \mathcal{X}_{p+1}$
- $\mathcal{N}_{p+1} = \widehat{\mathcal{X}}_{p+1} \cap L_0$

then $\widehat{\mathcal{X}}_{p+1}$ satisfies the conditions of Definitions 16.1, 16.2 and 16.3.

Proof. The set K_0 is a union of plaques of \mathcal{F} , which are convex subsets in the leafwise metric. The assumption that $K_0 - \text{Pen}_{\mathcal{F}}(\mathcal{N}_p, d_2'') \neq \emptyset$ then implies that there exists $\xi'_{p+1} \in K_0$ such that

$$(143) \quad B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200) \subset K_0 \cap \left\{ \text{Pen}_{\mathcal{F}}(\widehat{\mathcal{X}}_p, d_2'') - \text{Pen}_{\mathcal{F}}(\widehat{\mathcal{X}}_p, d_1'') \right\}$$

Then for all $z \in \widehat{\mathcal{X}}_p$ we have $d_{\mathcal{F}}(\xi'_{p+1}, z) > d_1''$. Choose $z_{p+1} \in \mathcal{M}_0 \cap B_{\mathcal{F}}(\xi'_{p+1}, \epsilon_{\mathcal{U}}^{\mathcal{F}}/2)$, which is possible by the assumption that \mathcal{M}_0 is a net which is $\epsilon_{\mathcal{U}}^{\mathcal{F}}/2$ -dense. Set $\theta_{p+1} = i_{z_{p+1}}$.

We next modify the choice of ξ'_{p+1} to a point $\xi_{p+1} \in B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ so that the conditions of the proposition for $\widehat{\mathcal{X}}_{p+1}$ are satisfied.

Consider the disk $D_{\mathcal{F}}(\xi'_{p+1}, 4d_2) \subset B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*) \subset B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}})$. Introduce the set

$$(144) \quad \Omega(\xi'_{p+1}) = D_{\mathcal{F}}(\xi'_{p+1}, 4d_2) \cap \widehat{\mathcal{X}}_p$$

Since the points of $\mathcal{N}_p(y_0)$ are d_1' -separated, and $d_2 = 2d_1$, the metric conditions (102), (103) and (104) and a standard volume estimate yields that the cardinality of $\Omega(\xi'_{p+1})$ is at most 10^n .

Let $\Omega^{(n)}(\xi'_{p+1}) \subset \Delta_{\mathcal{F}}^{(n)}(\widehat{\mathcal{X}}_p)$ be the subset of all n -simplices whose vertices are contained in $\Omega(\xi'_{p+1})$. The cardinality of the set $\Omega^{(n)}(\xi'_{p+1})$ is thus bounded above by the constant C_n defined in (108).

For each n -simplex $\Delta(y_0, \dots, y_n) \in \Omega^{(n)}(\xi'_{p+1})$ recall that $\omega(y_0, \dots, y_n) \in D_{\mathcal{F}}(\xi'_{p+1}, 4d_2)$ denotes the center of the inscribed sphere for its vertices, so $\{y_0, \dots, y_n\} \subset S_{\mathcal{F}}(\omega(y_0, \dots, y_n), r(y_0, \dots, y_n))$.

For each n -simplex $\Delta(y_0, \dots, y_n) \in \Omega^{(n)}(\xi'_{p+1})$ and constant $\kappa > 0$, form the annular region

$$A_{\mathcal{F}}(y_0, \dots, y_n; \kappa) = \text{Pen}_{\mathcal{F}}(S_{\mathcal{F}}(\omega(y_0, \dots, y_n), r(y_0, \dots, y_n)), \kappa)$$

LEMMA 17.3. *Let $\kappa = 2\varepsilon_1 \lambda_{\mathcal{F}}^*$. Then for an n -simplex $\Delta(y_0, \dots, y_n) \in \Omega^{(n)}(\xi'_{p+1})$,*

$$(145) \quad \text{Vol}_{\mathcal{F}} A_{\mathcal{F}}(y_0, \dots, y_n; \kappa) \leq 200 \cdot 2^n \varepsilon_1 (\lambda_{\mathcal{F}}^*/5)^n$$

Proof. Let Φ_n be the constant such that $\text{Vol}_{\widehat{u}} D_{\mathbb{R}^n}(s) = \Phi_n s^n$, where $\text{Vol}_{\widehat{u}}$ denotes the volume with respect to the frame \widehat{u} . Note that $(\sqrt{2})^n \leq \Phi_n \leq 2^n$.

By the condition (104) for $0 < s \leq \lambda_{\mathcal{F}}^* \leq \lambda_{\varepsilon_0}$ we have

$$|\Phi_n s^n - \text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\omega(y_0, \dots, y_n), s)| \leq \varepsilon_0 \cdot s^n \leq \varepsilon_0 \cdot (\lambda_{\mathcal{F}}^*)^n$$

Hence, we have

$$\begin{aligned} \text{Vol}_{\mathcal{F}} A_{\mathcal{F}}(y_0, \dots, y_n; \kappa) &= |\text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\omega(y_0, \dots, y_n), r(y_0, \dots, y_n) + \kappa) - \text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\omega(y_0, \dots, y_n), r(y_0, \dots, y_n) - \kappa)| \\ &\leq \Phi_n \cdot \{(r(y_0, \dots, y_n) + \kappa)^n - (r(y_0, \dots, y_n) - \kappa)^n\} + 2\varepsilon_0 (\lambda_{\mathcal{F}}^*)^n \end{aligned}$$

Given $\kappa = 2\varepsilon_1 \lambda_{\mathcal{F}}^*$ with $20n\varepsilon_1 < 1$, and $\lambda_{\mathcal{F}}^*/10 \leq r \leq \lambda_{\mathcal{F}}^*/5$, elementary estimates yield

$$\begin{aligned} \{(r + \kappa)^n - (r - \kappa)^n\} &= r^n \cdot \{(1 + \kappa/r)^n - (1 - \kappa/r)^n\} \\ &\leq r^n \cdot \{\exp(n\kappa/r) - \exp(-n\kappa/r)\} \\ &\leq (\lambda_{\mathcal{F}}^*/5)^n \cdot \{\exp(20n\varepsilon_1) - \exp(-20n\varepsilon_1)\} \\ &\leq 100n\varepsilon_1 (\lambda_{\mathcal{F}}^*/5)^n \end{aligned}$$

Combining these estimates and condition 2 in (116), we obtain

$$\begin{aligned} \text{Vol}_{\mathcal{F}} A_{\mathcal{F}}(y_0, \dots, y_n, \kappa) &\leq \Phi_n \cdot \{(r(y_0, \dots, y_n) + \kappa)^n - (r(y_0, \dots, y_n) - \kappa)^n\} + 2\varepsilon_0 (\lambda_{\mathcal{F}}^*)^n \\ &\leq \{\Phi_n \cdot 100n\varepsilon_1 + 2 \cdot 5^n \varepsilon_0\} (\lambda_{\mathcal{F}}^*/5)^n \\ &\leq \{2^n \cdot 100n\varepsilon_1 + 2 \cdot 5^n \varepsilon_0\} (\lambda_{\mathcal{F}}^*/5)^n \\ (146) \quad &\leq 200n \cdot 2^n \varepsilon_1 (\lambda_{\mathcal{F}}^*/5)^n \quad \square \end{aligned}$$

The total volume of all such annular regions intersecting $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/100)$ is bounded above by

$$C_n \cdot 200n \cdot 2^n \varepsilon_1 (\lambda_{\mathcal{F}}^*/5)^n$$

We also have the estimate of the leafwise volume of the disk

$$|\Phi_n \cdot (\lambda_{\mathcal{F}}^*/200)^n - \text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)| \leq \varepsilon_0 \cdot (\lambda_{\mathcal{F}}^*/200)^n = (\varepsilon_0/2^n) \cdot (\lambda_{\mathcal{F}}^*/100)^n$$

so that

$$(147) \quad \text{Vol}_{\mathcal{F}} D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200) \geq \Phi_n \cdot (\lambda_{\mathcal{F}}^*/200)^n - \varepsilon_0 \cdot (\lambda_{\mathcal{F}}^*/200)^n \geq (1/40)^n \cdot (\lambda_{\mathcal{F}}^*/5)^n$$

Now, given $\varepsilon_1 = 1/(C_n \cdot 1000n \cdot 100^n)$ by (109), it follows that

$$(148) \quad C_n \cdot 200n \cdot 2^n \varepsilon_1 \cdot (\lambda_{\mathcal{F}}^*/5)^n \leq \frac{1}{4} \cdot 1/40^n \cdot (\lambda_{\mathcal{F}}^*/5)^n$$

Thus, the total volume of all annular regions intersecting $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ is less than $1/4$ of its volume. Therefore, if we choose $\xi_{p+1} \in B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200) \subset K_0$ which lies outside of the union of these annular regions, then it will satisfy a stronger estimate than (141), though only for L_0 :

$$(149) \quad \text{For all } \Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{N}_p) \\ d_{\mathcal{F}}(\xi_{p+1}, \omega(y_0, \dots, y_n)) \geq r(y_0, \dots, y_n) + 2\varepsilon_1 \lambda_{\mathcal{F}}^*$$

The estimate (149) for the leaf L_0 implies that (141) holds for $\mathcal{N}_{p+1}(x)$ for all $x \in \mathfrak{M}$. This requires that we modify the choice of ξ_{p+1} to also satisfy the robustness condition Definition 15.6 as follows.

The idea is to again use volume estimates, in this case for the sets of points for which the robustness condition fails, then chose ξ_{p+1} in the complement. The method is primitive, but it works.

For $1 \leq k < n$, let $\{y_0, \dots, y_k\} \subset \Omega(\xi'_{p+1})$ be a collection of distinct points with $y_k \in \mathcal{X}_{i_k}$ where $i_0 < \dots < i_k \leq p$.

Let $\hat{u} = \{\vec{u}_1, \dots, \vec{u}_n\} \subset T_{y_k} \mathcal{F}$ be an orthonormal frame, and $\psi_{y_k, \hat{u}}^g: D(\lambda_{\mathcal{F}}^*) \rightarrow D_{\mathcal{F}}(y_k, \lambda_{\mathcal{F}}^*) \subset L_0$ the corresponding geodesic coordinates. Define $\vec{y}_j = (\psi_{y_k, \hat{u}}^g)^{-1}(y_j)$ for $0 \leq j \leq k+1$. Note that $\vec{y}_k = \vec{0}$.

Let $\text{Span}(\vec{y}_0, \dots, \vec{y}_k) \subset \mathbb{R}^n$ be the linear submanifold through the origin of dimension k which they span. Then define a submanifold of $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$,

$$(150) \quad H(y_0, \dots, y_k; \xi'_{p+1}) = \psi_{y_k, \hat{u}}^g \{ \text{Span}(\vec{y}_0, \dots, \vec{y}_k) \cap D(2d_2) \} \cap D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$$

which has diameter at most $\lambda_{\mathcal{F}}^*/100$, and thus has $(n-1)$ -volume bounded above by $(\lambda_{\mathcal{F}}^*/100)^{n-1}$.

Form the $2\varepsilon_2 \lambda_{\mathcal{F}}^*$ -thickening of $H(y_0, \dots, y_k; \xi'_{p+1})$,

$$(151) \quad \mathcal{S}(y_0, \dots, y_k; \xi'_{p+1}, 2\varepsilon_2 \lambda_{\mathcal{F}}^*) = \text{Pen}_{\mathcal{F}}(H(y_0, \dots, y_k; \xi'_{p+1}), 2\varepsilon_2 \lambda_{\mathcal{F}}^*) \cap D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$$

Then by the estimate (105) and $\varepsilon_0 \leq \varepsilon_2$, its volume is bounded above by

$$(152) \quad 4(\varepsilon_2 \lambda_{\mathcal{F}}^*) \cdot (\lambda_{\mathcal{F}}^*/100)^{n-1} + \varepsilon_0 (\lambda_{\mathcal{F}}^*/100)^n \leq 5(\varepsilon_2 \lambda_{\mathcal{F}}^*) \cdot (\lambda_{\mathcal{F}}^*/100)^{n-1}$$

The total number of such submanifolds $H(y_0, \dots, y_k; \xi'_{p+1})$ in $D_{\mathcal{F}}(\xi'_{p+1}, 4d_2)$ is bounded above by the constant C_n from (108), hence the total volume of all such sets which intersect $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ is thus bounded above by

$$(153) \quad C_n \cdot 5(\varepsilon_2 \lambda_{\mathcal{F}}^*) \cdot (\lambda_{\mathcal{F}}^*/100)^{n-1} = \frac{C_n \cdot (5\lambda_{\mathcal{F}}^*) \cdot (\lambda_{\mathcal{F}}^*/100)^{n-1}}{C_n \cdot 2000 \cdot 2^n} = \frac{1}{4} \cdot 1/40^n \cdot (\lambda_{\mathcal{F}}^*/5)^n$$

where we use the definition of ε_2 in (110).

Thus, the total volume of all slabs intersecting $D_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ is less than $1/4$ of its volume, so we may choose $\xi_{p+1} \in B_{\mathcal{F}}(\xi'_{p+1}, \lambda_{\mathcal{F}}^*/200)$ which is disjoint from the union of all annular and slab regions introduced above. This completes the choice of the new point ξ_{p+1} .

First, we note that $\hat{\mathcal{X}}_{p+1}$ is d'_1 -separated.

LEMMA 17.4. *For all $y \neq z \in \widehat{\mathcal{X}}_{p+1}$ we have $d_{\mathcal{F}}(y, z) \geq .114 \cdot \lambda_{\mathcal{F}}^* > d'_1$.*

Proof. If y, z lie on distinct leaves, there is nothing to show. Assume $y \neq z \in \mathcal{N}_{p+1}(x) = \widehat{\mathcal{X}}_{p+1} \cap L_x$. Then by definition, there exists $\xi_i, \xi_j \in \widehat{\mathcal{X}}_{p+1}$ for $1 \leq i, j \leq p+1$ such that $y \in \mathfrak{Z}(\xi_i, \theta_i, V_0) \cap L_x$ and $z \in \mathfrak{Z}(\xi_j, \theta_j, V_0) \cap L_x$. Without loss of generality we can assume that $i \geq j$.

If $z \notin \mathcal{P}_{\theta_i}(y)$ then $d_{\mathcal{F}}(y, z) \geq \delta_{\mathcal{U}}^{\mathcal{F}} > \lambda_{\mathcal{F}}^* > d'_1$. Thus, we may assume that $z \in \mathcal{P}_{\theta_i}(y)$, so $i > j$.

Set $y' = \mathfrak{Z}(\xi_i, \theta_i, V_0) \cap \mathcal{P}_{\theta_i}(\xi_i) = \xi_i$ and set $z' = \mathfrak{Z}(\xi_j, \theta_j, V_0) \cap \mathcal{P}_{\theta_i}(\xi_i)$.

Then $d_{\mathcal{F}}(y', z') \geq d''_1 - \lambda_{\mathcal{F}}^*/200 = .115 \cdot \lambda_{\mathcal{F}}^*$ by the choice of ξ'_i satisfying (143) and the choice of ξ_i .

Apply Lemma 15.3 for the pairs $\{y, z\}$ and $\{y', z'\}$ and note that $2\varepsilon_0 < 1/1000$ to obtain

$$(154) \quad d_{\mathcal{F}}(y, z) \geq d_{\mathcal{F}}(y', z') - 2\varepsilon_0 \cdot \lambda_{\mathcal{F}}^* > .115 \cdot \lambda_{\mathcal{F}}^* - .001 \cdot \lambda_{\mathcal{F}}^* = .114 \cdot \lambda_{\mathcal{F}}^*$$

Thus, for all $x \in \mathfrak{M}$ the net $\mathcal{N}_{p+1}(x)$ is $(.114 \cdot \lambda_{\mathcal{F}}^*)$ -separated. \square

A simple consequence of Lemma 17.4 is that if $\widehat{\mathcal{X}}_p$ is uniformly d'_1 -separated, then the collection of leafwise disks $\{D_{\mathcal{F}}(\xi_\ell, d'_1/2) \mid \xi_\ell \in \Xi_p\}$ are pairwise disjoint. As the set K_0 was assumed to be compact, this implies the cardinality of the set Ξ_p has an a priori bound. That is, we can repeat the construction in Proposition 17.2 at most a finite number of times, until we obtain a regular partial V_0 -transversal $\widehat{\mathcal{X}}_{p_*}$ for K_0 which is d''_2 -complete, for some $p_* > 0$.

Set $\mathcal{X} = \widehat{\mathcal{X}}_{p_*}$, set $\mathcal{N} = \mathcal{X} \cap L_0$ and for $x \in \mathfrak{M}$ let $\mathcal{N}(x) = \mathcal{X} \cap L_x$.

The set \mathcal{X} is d'_1 -separated by Lemma 17.4. It is d'_2 -dense in $\mathfrak{N}(K_0, V_0)$ by the following:

LEMMA 17.5. *For $y \in \mathfrak{N}(K_0, V_0)$ there exists $z \in \mathcal{N}(y)$ such that $d_{\mathcal{F}}(y, z) \leq .181 \cdot \lambda_{\mathcal{F}}^* < d'_2$.*

Proof. Let $y \in \mathfrak{N}(K_0, V_0)$. The by the constructions in Section 16, there exists $z \in \mathcal{M}_{K_0}$ for which $y = \mathcal{P}_{i_z}(y) \cap \mathfrak{Z}(y, i_z, V_0)$. Choose $\zeta \in \mathfrak{U}_z^{V_0} \cap K_0$ and set $y' = \mathcal{P}_{i_z}(\zeta) \cap \mathfrak{Z}(y, i_z, V_0) \in K_0$.

We are given that $K_0 \subset \text{Pen}_{\mathcal{F}}(\mathcal{N}, d''_2)$, so there exists $\xi \in \mathcal{N}$ such that $d_{\mathcal{F}}(y', \xi) \leq d''_2 = .18 \cdot \lambda_{\mathcal{F}}^*$.

By definition of \mathcal{N} , there exists $\xi_j \in \Xi$ for some $1 \leq j \leq p_*$ such that $\xi = \mathcal{P}_{\theta_i}(\xi) \cap \mathfrak{Z}(\xi_j, \theta_i, V_0)$.

Let $z = \mathcal{P}_{\theta_i}(y) \cap \mathfrak{Z}(\xi_j, \theta_i, V_0) \in \mathcal{N}(y)$, and apply Lemma 15.3 for the pairs $\{y, z\}$ and $\{y', \xi\}$ to obtain

$$(155) \quad d_{\mathcal{F}}(y, z) \leq d_{\mathcal{F}}(y', \xi) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* \leq d''_2 + .001 \cdot \lambda_{\mathcal{F}}^* = .181 \cdot \lambda_{\mathcal{F}}^*$$

Thus \mathcal{X} is $.181 \cdot \lambda_{\mathcal{F}}^*$ -dense in $\mathfrak{N}(K_0, V_0)$. \square

We must show that \mathcal{X} satisfies the remaining conditions of Definitions 16.2 and 16.3. We first establish Definition 16.3 for \mathcal{X} , where it suffices to show that the robustness condition is stable. The proof uses an induction procedure which invokes Proposition 15.7 repeatedly, and the constants defined by (113) and the inequalities (114), (115) defining ε_4 . For $1 \leq \ell \leq n$, recall the definitions $\tilde{\rho}_\ell = \hat{\rho}_\ell \lambda_{\mathcal{F}}^*/10$ and $\tilde{\rho}'_\ell = \hat{\rho}'_\ell \lambda_{\mathcal{F}}^*/10$.

PROPOSITION 17.6. *Let $\rho = 3\varepsilon_2 \lambda_{\mathcal{F}}^*/2$. Then for each $\Delta(x_0, \dots, x_n) \in \Delta'_{\mathcal{F}}(\widehat{\mathcal{X}}_{p+1})$, such that the vertices $\{x_0, \dots, x_n\}$ are properly ordered, then the collection $\{x_0, \dots, x_n\}$ is ρ -robust.*

Proof. We proceed via induction on $1 \leq m < n$. We recall some notation. Let $(y_0, \dots, y_n) \subset L_0$ such that $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\widehat{\mathcal{X}}_{p+1})$. By permuting the order of the vertices, we can assume that the vertices $\{x_0, \dots, x_n\}$ are properly ordered. That is, there exists $1 \leq i_0 < i_1 < \dots < i_n \leq p+1$ and points $\xi_{i_k} \in \widehat{\Xi}_{p+1} \subset L_0$ such that $y_k = \mathcal{X}_{i_k} \cap \mathcal{P}_{\theta_{i_n}}(y_n)$. For notational convenience, set $\mathcal{P}_n(z) = \mathcal{P}_{\theta_{i_n}}(z)$.

The subtlety of the proof lies in the fact that the robust condition in Definition 15.6 is with respect to the geodesic coordinates about the last vertex in the collection properly ordered points, but the inductive hypotheses are in terms of the geodesic coordinates about previous vertices. The change of coordinates from one vertex to another introduces an error in the robust condition. Consequently,

at each stage of the induction, the robust constants $\tilde{\rho}_\ell$ decrease to account for this error. This fact is behind the arcane definition in formula (113).

The first step of the induction, $m = 1$, is trivial. Note that $\tilde{\rho}_0 = 18\varepsilon_2\lambda_{\mathcal{F}}^*/10$. Then given $\{x_0, x_1\} \subset \mathcal{P}_1(\xi_{i_1})$ as above, with $x_\ell \in \mathcal{X}_{i_\ell}$ and $i_0 \neq i_1$ then $d_{\mathcal{F}}(x_1, x_0) \geq d'_1$ by Lemma 17.4. Hence

$$d'_1 \geq 2\varepsilon_2\lambda_{\mathcal{F}}^* > \tilde{\rho}_0 > \tilde{\rho}_1$$

and so $\{x_0, x_1\}$ is $\tilde{\rho}_1$ -robust.

Now assume that $1 < m < n$. We make an inductive hypothesis which is uniform for all simplices. That is, for fixed $m < n$, assume that for all $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\hat{\mathcal{X}}_{p+1})$, then for all subsets of points $\{x_0, \dots, x_m\}$ defined for $x_n \in \mathcal{X}_{i_m}$ as above, the set $\{x_0, \dots, x_m\}$ is $\tilde{\rho}_m$ -robust. We then show that each transverse translate $\{x_0, \dots, x_{m+1}\}$ of $\{y_0, \dots, y_{m+1}\}$ is $\tilde{\rho}_{m+1}$ -robust.

Consider first the case $z_{m+1} = \xi_{i_{m+1}} \in \mathcal{X}_{i_{m+1}}$, and set $z_k = \mathcal{X}_{i_k} \cap \mathcal{P}_n(\xi_{i_{m+1}})$ for $0 \leq j \leq n$. By the inductive hypothesis, the set $\{z_0, \dots, z_m\}$ is $\tilde{\rho}_m$ -robust. We verify the conditions of Definition 15.6 for the vertex, z_{m+1} . The point $\xi_{i_{m+1}}$ was chosen to that it lies outside of all $2\varepsilon_2\lambda_{\mathcal{F}}^*$ -neighborhoods as defined in (151) of the images under the exponential map of affine subspaces spanned by local collections of at most $n + 1$ points. It follows, in particular, that the distance from $\xi_{i_{m+1}}$ to the submanifold $H(z_0, \dots, z_m; z_m)$ in Definition 15.6.3 is at least $2\varepsilon_2\lambda_{\mathcal{F}}^* > \tilde{\rho}_m$. Thus, $\{z_0, \dots, z_{m+1}\}$ is also $\tilde{\rho}_m$ -robust.

Note that $d'_1 \leq d_{\mathcal{F}}(z_j, z_k)$ for $0 \leq j \neq k \leq n$ by Lemma 17.4. We are given $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\hat{\mathcal{X}}_{p+1})$ which implies that the vertices $\{y_0, \dots, y_n\}$ admit an inscribed sphere, which must have radius at most $.181 \cdot \lambda_{\mathcal{F}}^*$ by Lemma 17.5. Thus, $d_{\mathcal{F}}(y_j, y_k) \leq .362 \cdot \lambda_{\mathcal{F}}^*$. The map ϕ_{y_n, z_n} is an $\varepsilon_0\lambda_{\mathcal{F}}^*$ -isometry by Lemma 15.3, so $d_{\mathcal{F}}(z_j, z_k) \leq .362 \cdot \lambda_{\mathcal{F}}^* + \varepsilon_0 \cdot \lambda_{\mathcal{F}}^* < .380 \cdot \lambda_{\mathcal{F}}^* = 2d'_2$. It follows that the set of points $\{z_0, \dots, z_n\}$ satisfy the hypotheses of Proposition 15.7 for $e_1 = d'_1$, $e_2 = d'_2$, and $\rho = \tilde{\rho}_m$.

For simplicity, set $\zeta = z_{m+1}$, and choose an orthonormal frame \hat{u} of $T_{\zeta}\mathcal{F}$. Then for $0 \leq j \leq m$, let $\tilde{z}_j = (\psi_{\zeta, \hat{u}}^g)^{-1}(z_j)$ for the geodesic coordinates $\psi_{\zeta, \hat{u}}^g: D(\lambda_{\mathcal{F}}^*) \rightarrow D_{\mathcal{F}}(\zeta, \lambda_{\mathcal{F}}^*)$. Then the collection $\{\tilde{z}_0, \dots, \tilde{z}_{m+1}\} \subset \mathbb{R}^n$ is $\tilde{\rho}'_m$ -robust by Proposition 15.7 and the choice of ε_4 in (114) and (115).

The robustness for the set $\{\tilde{z}_0, \dots, \tilde{z}_{m+1}\}$ is used to show it for $\{x_0, \dots, x_{m+1}\}$. Set $\zeta' = x_{m+1}$, then by Proposition 15.4, there exists an orthonormal framing \hat{v} of $T_{\zeta'}\mathcal{F}$ so that the composition

$$\Psi_{\zeta, \zeta'} \equiv (\psi_{\zeta', \hat{v}}^g)^{-1} \circ \phi_i(\zeta, \zeta') \circ \psi_{\zeta, \hat{u}}^g \circ T_{\zeta}: D(\lambda_{\mathcal{F}}^*/2) \rightarrow \mathbb{R}^n \cong T_{\zeta'}\mathcal{F}$$

is $\varepsilon_4\lambda_{\mathcal{F}}^*$ -close to the identity. Set $\vec{w}_j = (\psi_{\zeta', \hat{v}}^g)^{-1}(x_k) = \Psi_{\zeta, \zeta'}(\tilde{z}_j)$, then $\|\vec{w}_j - \tilde{z}_j\| \leq \varepsilon_4\lambda_{\mathcal{F}}^*$.

The set $\{\vec{w}_1, \dots, \vec{w}_{m+1}\}$ satisfies the hypotheses of Proposition 12.3 for $e_1 = d_1$, $e_2 = d_2$, $\varepsilon = \varepsilon_4\lambda_{\mathcal{F}}^*$ and $\rho = \tilde{\rho}'_m$. Therefore, $\{\vec{w}_1, \dots, \vec{w}_{m+1}\}$ is $(\tilde{\rho}_{m+1} + \varepsilon_2\lambda_{\mathcal{F}}^*/1000)$ -robust.

Finally, by Lemma 15.2 the geodesic map $\psi_{\zeta', \hat{v}}^g$ is an $\varepsilon_0\lambda_{\mathcal{F}}^*$ -isometry, hence the distance from x_{m+1} to the submanifold $H(x_0, \dots, x_m; x_m)$ in Definition 15.6.3 is at least $\tilde{\rho}_{m+1} + \varepsilon_2\lambda_{\mathcal{F}}^*/1000 - \varepsilon_0\lambda_{\mathcal{F}}^* > \tilde{\rho}_{m+1}$.

This completes the inductive step. It remains to note that $\tilde{\rho}_n > 3\varepsilon_2\lambda_{\mathcal{F}}^*/2$ by definition (113). \square

Finally, the stability conditions of Definition 16.2 for \mathcal{X} follow from the results of the next section.

18. STABILITY

It remains to show that the transversal \mathcal{X} is regular and stable. At first inspection, stability of simplices for a Delaunay triangulation associated to a leafwise net $\mathcal{N}(x)$ for $x \in \mathcal{X}$ appears to be “intuitively clear”, and in fact this is basically correct for dimension $n \leq 2$. The difficulty is that for $n > 2$, as x varies, the “small variations” of the points of $\mathcal{N}(x)$ may result in an abrupt change in the Delaunay simplicial structure, if any face of a Voronoi cell has too small of a diameter relative to the size of the variation. Proposition 17.6 along with Propositions 18.1 and 18.4 which follow, show this does not happen for the nets defined by the section \mathcal{X} .

We first establish some notation used in the demonstrations. Let $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$. By permuting the order of the vertices, we can assume that there exists $1 \leq i_0 < i_1 < \dots < i_n \leq p_*$ and points $\xi_{i_k} \in \Xi_{p_*} \subset L_0$ such that $y_k = \mathcal{X}_{i_k} \cap \mathcal{P}_{\theta_{i_n}}(y_n)$.

For $x_n \in \mathcal{X}_{i_n} \subset \mathcal{U}_{\theta_{i_n}}^{V_0}$ let $\mathcal{P}_n(x_n) = \mathcal{P}_{\theta_{i_n}}(x_n)$ denote the plaque containing x_n in the chart φ_{i_n} .

Set $x_k = \mathcal{X}_{i_k} \cap \mathcal{P}_n(x_n)$ for $0 \leq k \leq n$.

The next step towards showing that \mathcal{X} is regular and stable is to show that (x_0, \dots, x_n) admits an inscribed sphere, satisfying certain restrictions, so that $\Delta(x_0, \dots, x_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$.

18.1. Constructing inscribed spheres.

PROPOSITION 18.1. *For $x_n \in \mathcal{X}_{i_n}$ with (x_0, \dots, x_n) defined as above, there exists $r(x_0, \dots, x_n)$ and $\omega(x_0, \dots, x_n) \in \mathcal{P}_n(x_n)$ such that*

$$(156) \quad \{x_0, \dots, x_n\} \subset S_{\mathcal{F}}(\omega(x_0, \dots, x_n), r(x_0, \dots, x_n)) \cap \mathcal{N}(x_n)$$

Moreover, the center satisfies, for ε_3 defined by (111),

$$(157) \quad d_{\mathcal{F}}(\omega(x_0, \dots, x_n), \omega'(y_0, \dots, y_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

where $\omega'(y_0, \dots, y_n) = \phi_{i_n}(y_n, x_n)(\omega(y_0, \dots, y_n))$ is the translate for the center of the inscribed sphere for the n -simplex $\Delta(y_0, \dots, y_n) \in \Delta_{\mathcal{F}}^{(n)}(\mathcal{X})$. In particular, this implies

$$(158) \quad |r(x_0, \dots, x_n) - r(y_0, \dots, y_n)| < \varepsilon_3 \lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* < \varepsilon_3 \lambda_{\mathcal{F}}^*$$

Proof. By rearranging the order of the vertices if necessary, we may assume that there are indices $i_0 < i_1 < \dots < i_n \leq p_*$ and points $\xi_{i_k} \in \Xi_{p_*} \subset L_0$ such that $y_k = \mathcal{X}_{i_k} \cap \mathcal{P}_n(y_n)$ for $0 \leq k \leq n$.

Let $\omega = \omega(y_0, \dots, y_n) \in \mathcal{P}_n(y_n)$ denote the center of the inscribed sphere for $\{y_0, \dots, y_n\}$, and let $r(y_0, \dots, y_n)$ denote its radius. Then $d'_1/2 \leq r(y_0, \dots, y_n) \leq d'_2$ as $\mathcal{N}(y_n)$ is d'_2 -dense and d'_1 -separated. Note that this implies $\{y_0, \dots, y_n, \omega\} \subset D(y_n, \lambda_{\mathcal{F}}^*/5)$.

By Proposition 17.6, the set $\{x_0, \dots, x_n\} \subset \mathcal{P}_n(x_n)$ is $\tilde{\rho}_n$ -robust.

Let $\phi_{i_n}(y_n, x_n): \mathcal{P}_n(y_n) \rightarrow \mathcal{P}_n(x_n)$ be the transverse transport map for the chart φ_{i_n} .

Let $\omega'(y_0, \dots, y_n) = \phi_{i_n}(y_n, x_n)(\omega)$ denote the translation of $\omega(y_0, \dots, y_n)$ to $\mathcal{P}_n(x_n)$.

For $0 \leq j \leq n$, we have the radius equalities $d_{\mathcal{F}}(y_j, \omega) = r(y_0, \dots, y_n)$, hence by Lemma 15.3,

$$(159) \quad r(y_0, \dots, y_n) - 2\varepsilon_0 \lambda_{\mathcal{F}}^* \leq d_{\mathcal{F}}(x_j, \omega'(y_0, \dots, y_n)) \leq r(y_0, \dots, y_n) + 2\varepsilon_0 \lambda_{\mathcal{F}}^*$$

Indeed, note that x_ℓ is the transverse transport of y_ℓ for the coordinate system φ_{i_ℓ} , while $\omega'(y_0, \dots, y_n)$ is the transport of $\omega(y_0, \dots, y_n)$ for the coordinate system φ_{i_n} and $i_\ell \neq i_n$. Thus, we must use (119) in place of the sharper estimate (120). Similarly, for $0 \leq j \neq k \leq n$, we have

$$(160) \quad d'_1 \leq d_{\mathcal{F}}(x_j, x_k) \leq 2d'_2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* < 2d_2$$

It follows that we also have $\{x_0, \dots, x_n, \omega'(y_0, \dots, y_n)\} \subset D(x_n, \lambda_{\mathcal{F}}^*/5)$.

The first step is to construct an inscribed sphere with center $\vec{\omega}(\vec{v}_0, \dots, \vec{v}_n)$ for the linearized problem in the tangent space $T_{x_n}\mathcal{F}$, and then modify the construction to obtain an inscribed sphere with center $\omega(x_0, \dots, x_n) \in \mathcal{P}_n(x_n)$ for the leafwise metric.

Choose $\xi \in \mathcal{P}_n(x_n)$ so that $\{x_0, \dots, x_n, \omega'(y_0, \dots, y_n)\} \subset B_{\mathcal{F}}(\xi, 2d_2)$. Let $\hat{u} = \{\vec{u}_1, \dots, \vec{u}_n\} \subset T_{\xi}\mathcal{F}$ be an orthonormal frame, with corresponding geodesic coordinates $\psi_{\xi, \hat{u}}^g$ about ξ .

Set $\vec{v}_k = (\psi_{\xi, \hat{u}}^g)^{-1}(x_k)$ for $0 \leq k \leq n$, then $\{\vec{v}_0, \dots, \vec{v}_n\} \subset \mathbb{R}^n$ is $\tilde{\rho}'_{n+1}$ -robust by Proposition 15.7.

Now set $\vec{\omega}'(y_0, \dots, y_n) = (\psi_{\xi, \hat{u}}^g)^{-1}(\omega'(y_0, \dots, y_n))$. Then by Lemma 15.2 and (159) we have

$$(161) \quad r(y_0, \dots, y_n) - 3\varepsilon_0 \lambda_{\mathcal{F}}^* \leq \|\vec{v}_j - \vec{\omega}'(y_0, \dots, y_n)\| \leq r(y_0, \dots, y_n) + 3\varepsilon_0 \lambda_{\mathcal{F}}^*$$

while Lemma 15.2 and (160) implies, for $0 \leq j \neq k \leq n$,

$$(162) \quad d_1 < d'_1 - \varepsilon_0 \lambda_{\mathcal{F}}^* \leq \|\vec{v}_j - \vec{v}_k\| \leq d'_2 + 3\varepsilon_0 \lambda_{\mathcal{F}}^* < d_2$$

We can thus apply Proposition 12.2 for $e_1 = d_1$, $e_2 = d_2 = 2d_1$, $\rho = \tilde{\rho}'_{n+1} > 3\varepsilon_2\lambda_{\mathcal{F}}^*/2$ and $C_1 = 3\varepsilon_0\lambda_{\mathcal{F}}^*$ to conclude that there exists an inscribed sphere $S(\omega(\vec{v}_0, \dots, \vec{v}_n), r(\vec{v}_0, \dots, \vec{v}_n)) \subset D(\lambda_{\mathcal{F}}^*)$ such that

$$\begin{aligned} \|\omega(\vec{v}_0, \dots, \vec{v}_n) - \vec{\omega}'(y_0, \dots, y_n)\| &\leq 3\varepsilon_0 \cdot \left\{ n^{3/2} (2d_2)^{n-1} / (3\varepsilon_2\lambda_{\mathcal{F}}^*/2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^* \\ (163) \qquad \qquad \qquad &\leq 3\varepsilon_0 \cdot \left\{ n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^* \end{aligned}$$

That is, the vector $\omega(\vec{v}_0, \dots, \vec{v}_n)$ is a solution of the linearized problem of finding the center of an inscribed sphere, and (163) estimates the Euclidean distance to the translated center.

The task now is to convert this approximate answer to a solution for the leafwise metric. As before, let \tilde{d} denote the distance function on $D(\lambda_{\mathcal{F}}^*)$ induced from $d_{\mathcal{F}}$ by $\psi_{\xi, \hat{a}}^g: D(\lambda_{\mathcal{F}}^*/2) \rightarrow L_{\xi}$. Then by Lemma 15.2,

$$(164) \qquad \qquad \qquad |\tilde{d}(\vec{a}, \vec{b}) - \|\vec{a} - \vec{b}\|| \leq \varepsilon_0 \lambda_{\mathcal{F}}^* \quad , \quad \text{for all } \vec{a}, \vec{b} \in D(\lambda_{\mathcal{F}}^*/2)$$

Introduce the equidistant submanifolds for the metric \tilde{d} ,

$$(165) \qquad \qquad \qquad \mathcal{H}(\vec{v}_j, \vec{v}_k) = \{\vec{z} \in D(\lambda_{\mathcal{F}}^*/2) \mid \tilde{d}(\vec{z}, \vec{v}_j) = \tilde{d}(\vec{z}, \vec{v}_k)\}$$

and the “thickened” equidistant sets for the leafwise metric, for $\epsilon > 0$,

$$(166) \qquad \qquad \qquad \mathcal{H}(\vec{v}_j, \vec{v}_k; \epsilon) = \{\vec{z} \in D(\lambda_{\mathcal{F}}^*/2) \mid -\epsilon \leq \tilde{d}(\vec{z}, \vec{v}_j) - \tilde{d}(\vec{z}, \vec{v}_k) \leq \epsilon\}$$

$$(167) \qquad \qquad \qquad \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; \epsilon) = \mathcal{H}(\vec{v}_0, \vec{v}_n; \epsilon) \cap \dots \cap \mathcal{H}(\vec{v}_{n-1}, \vec{v}_n; \epsilon)$$

Then (159) implies the translation $\vec{\omega}'(y_0, \dots, y_n) \in \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*)$, so this set is not empty. The key idea is to obtain a bound for its diameter, from which the proof of Proposition 18.1 follows. To this end, define the set of approximate solutions of the linearized problem by

$$(168) \qquad \qquad B(\vec{v}_0, \dots, \vec{v}_n; \epsilon) = \{\vec{z} \in D(\lambda_{\mathcal{F}}^*/2) \mid -\epsilon \leq \|\vec{z} - \vec{v}_j\| - \|\vec{z} - \vec{v}_n\| \leq \epsilon, \quad 0 \leq j < n\}$$

Note that the actual solution satisfies $\omega(\vec{v}_0, \dots, \vec{v}_n) \in B(\vec{v}_0, \dots, \vec{v}_n; \epsilon)$ for all $\epsilon > 0$.

LEMMA 18.2. $\mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; \epsilon) \subset B(\vec{v}_0, \dots, \vec{v}_n; \epsilon + 4\varepsilon_0\lambda_{\mathcal{F}}^*)$

Proof. Using (164) for $\vec{z} \in D(\lambda_{\mathcal{F}}^*/2)$, we have that

$$(169) \quad |(\|\vec{z} - \vec{v}_j\| - \|\vec{z} - \vec{v}_k\|) - 2\varepsilon_0\lambda_{\mathcal{F}}^*| \leq |\tilde{d}(\vec{z}, \vec{v}_j) - \tilde{d}(\vec{z}, \vec{v}_k)| \leq |(\|\vec{z} - \vec{v}_j\| - \|\vec{z} - \vec{v}_k\|)| + 2\varepsilon_0\lambda_{\mathcal{F}}^*$$

and the claim follows. \square

Thus, we now have $\vec{\omega}'(y_0, \dots, y_n) \in \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*) \subset B(\vec{v}_0, \dots, \vec{v}_n; 8\varepsilon_0\lambda_{\mathcal{F}}^*)$.

LEMMA 18.3. *Let $\vec{z} \in B(\vec{v}_0, \dots, \vec{v}_n; 8\varepsilon_0\lambda_{\mathcal{F}}^*)$, then*

$$(170) \qquad \qquad \qquad \|\vec{z} - \omega(\vec{v}_0, \dots, \vec{v}_n)\| \leq 32\varepsilon_0 \cdot \left\{ n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^*$$

Proof. Using the notation of Propositions 12.1 and 12.2, with \vec{v}_j in place of \vec{z}_j and \vec{z} in place of ω , and $e_1 = d_1$, $e_2 = d_2 = 2d_1$, $\rho = \tilde{\rho}'_{n+1} > 3\varepsilon_2\lambda_{\mathcal{F}}^*/2$ and $C_1 = 8\varepsilon_0\lambda_{\mathcal{F}}^*$, then $\vec{\zeta} = \vec{z} - \omega(\vec{v}_0, \dots, \vec{v}_n)$ is a solution of the matrix inequality

$$(171) \qquad \qquad \qquad \mathbf{V} \cdot \vec{\zeta} \in B(0, 2\sqrt{n} \cdot d_2 \cdot 8\varepsilon_0\lambda_{\mathcal{F}}^*)$$

Then by (79), we have the estimate $\|\mathbf{V}^{-1}\| \leq n \cdot (2d_2)^{n-1} / d_1 (3\varepsilon_2\lambda_{\mathcal{F}}^*/2)^{n-1}$ which yields

$$\begin{aligned} \|\vec{z} - \omega(\vec{v}_0, \dots, \vec{v}_n)\| &\leq \left\{ n \cdot (2d_2)^{n-1} / d_1 (3\varepsilon_2\lambda_{\mathcal{F}}^*/2)^{n-1} \right\} \cdot \{2\sqrt{n} \cdot d_2 \cdot 8\varepsilon_0\lambda_{\mathcal{F}}^*\} \\ &\leq \varepsilon_0 \cdot \left\{ 32n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^* \end{aligned}$$

where we use that $d_1 = \lambda_{\mathcal{F}}^*/10$ and $d_2 = 2\lambda_{\mathcal{F}}^*/10$ to simplify, yielding (170). \square

It follows from Lemmas 18.2 and 18.3 that the closed set $\mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*)$ is bounded, and is non-empty as $\vec{\omega}'(y_0, \dots, y_n) \in \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*)$. Thus, the intersection

$$(172) \quad \tilde{\omega}(\vec{v}_0, \dots, \vec{v}_n) = \mathcal{H}(\vec{v}_0, \vec{v}_n) \cap \dots \cap \mathcal{H}(\vec{v}_{n-1}, \vec{v}_n) \subset \mathcal{B}(\vec{v}_0, \dots, \vec{v}_n; 4\varepsilon_0\lambda_{\mathcal{F}}^*)$$

is non-empty by transversality of the submanifolds $\mathcal{H}(\vec{v}_j, \vec{v}_n)$. Moreover, (170) implies that

$$(173) \quad \|\tilde{\omega}(\vec{v}_0, \dots, \vec{v}_n) - \omega(\vec{v}_0, \dots, \vec{v}_n)\| \leq 32\varepsilon_0 \cdot \left\{ n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^*$$

Combine this with the estimate (163) to obtain

$$(174) \quad \|\tilde{\omega}(\vec{v}_0, \dots, \vec{v}_n) - \vec{\omega}'(y_0, \dots, y_n)\| \leq 35\varepsilon_0 \cdot \left\{ n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^*$$

Then set

$$(175) \quad \omega(x_0, \dots, x_n) = \psi_{\xi, \hat{u}}^g(\tilde{\omega}(\vec{v}_0, \dots, \vec{v}_n)), \quad r(x_0, \dots, x_n) = d_{\mathcal{F}}(x_0, \omega(x_0, \dots, x_n))$$

so we have $\{x_0, \dots, x_n\} \subset S_{\mathcal{F}}(\omega(x_0, \dots, x_n), r(x_0, \dots, x_n))$ as desired.

Recall that $\omega'(y_0, \dots, y_n) = \psi_{\xi, \hat{u}}^g(\vec{\omega}'(y_0, \dots, y_n))$, then by Lemma 15.2 we have

$$(176) \quad d_{\mathcal{F}}(\omega(x_0, \dots, x_n), \omega'(y_0, \dots, y_n)) \leq \varepsilon_0 \cdot \left\{ 1 + 35n^{3/2} \cdot (4/15\varepsilon_2)^{n-1} \right\} \cdot \lambda_{\mathcal{F}}^* < \varepsilon_3\lambda_{\mathcal{F}}^*/2$$

where the bound by $\varepsilon_3\lambda_{\mathcal{F}}^*/2$ follows from (116).

Finally, the estimate (158) follows from

$$\begin{aligned} |r(x_0, \dots, x_n) - r(y_0, \dots, y_n)| &= |d_{\mathcal{F}}(x_n, \omega(x_0, \dots, x_n)) - d_{\mathcal{F}}(y_n, \omega(y_0, \dots, y_n))| \\ &\leq |d_{\mathcal{F}}(x_n, \omega(x_0, \dots, x_n)) - d_{\mathcal{F}}(x_n, \omega'(y_0, \dots, y_n))| + 2\varepsilon_0\lambda_{\mathcal{F}}^* \\ &\leq |d_{\mathcal{F}}(\omega(x_0, \dots, x_n), \omega'(y_0, \dots, y_n))| + 2\varepsilon_0\lambda_{\mathcal{F}}^* \\ &\leq \varepsilon_3\lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0\lambda_{\mathcal{F}}^* < \varepsilon_3\lambda_{\mathcal{F}}^* \end{aligned}$$

This completes the proof of Proposition 18.1. \square

18.2. Stability of the Delaunay simplicial complex. We have now established that for a simplex $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$, if $\{x_0, \dots, x_n\}$ is a transverse translate of the set $\{y_0, \dots, y_n\}$, then $\{x_0, \dots, x_n\}$ is $\tilde{\rho}_n > 3\varepsilon_2/2$ robust and admits an inscribed sphere whose radius varies according to the estimate (157). It remains to show that \mathcal{X} is stable, that is, $\Delta(x_0, \dots, x_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$. The only ingredient left to show is that the inscribed sphere for the set $\{x_0, \dots, x_n\}$ does not contain other points of \mathcal{X} in its interior.

PROPOSITION 18.4. *Let $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$. Assume given $1 \leq i_0 < i_1 < \dots < i_n \leq p_*$ and $\xi_{i_k} \in \Xi_{p_*} \subset L_0$ such that $y_k = \mathcal{X}_{i_k} \cap \mathcal{P}_{\theta_{i_n}}(y_n)$. Then for all $x_n \in \mathcal{X}_{i_n}$, $\Delta(x_0, \dots, x_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$.*

Proof. Let $\omega(y_0, \dots, y_n) \in \mathcal{P}_n(y_n)$ be the center of the inscribed sphere of radius $r(y_0, \dots, y_n)$. Then it is given that for all $\xi \in \mathcal{N}(y_n) - \{y_0, \dots, y_n\}$ we have that $d_{\mathcal{F}}(\xi, \omega(y_0, \dots, y_n)) > r(y_0, \dots, y_n)$.

We must show that the inscribed sphere for the set $\{x_0, \dots, x_n\}$ obtained in Proposition 18.1, with center $\omega(x_0, \dots, x_n)$ and radius $r(x_0, \dots, x_n)$, contains no points of \mathcal{X} in its interior. That is, we must show that

$$(177) \quad d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) > r(x_0, \dots, x_n) \quad \text{for all } \xi' \in \mathcal{N}(x_n) - \{x_0, \dots, x_n\}$$

Let $n \leq m \leq p_*$ be the largest m such that the condition (177) holds for all $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$ with $i_n \leq m$. If $m = p_*$ then we are done, so assume that $m < p_*$ and we show this leads to a contradiction. So we assume that we are given a simplex $\Delta(y_0, \dots, y_n)$ with $i_n = m + 1$, such that there is some $x_n \in \mathcal{X}_{i_n}$ and $\xi' \in \mathcal{N}(x_n) - \{x_0, \dots, x_n\}$ for which (177) fails.

First, consider the case where there exists $\xi' \in \mathcal{N}(x_n) - \{x_0, \dots, x_n\}$ such that

$$(178) \quad d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) \leq r(x_0, \dots, x_n) - 2\varepsilon_3\lambda_{\mathcal{F}}^*$$

Let $1 \leq q \leq p_*$ be such that $\xi' \in \mathcal{X}_q$ and set $\xi = \mathfrak{Z}(\xi', \theta_{i_q}, V_0) \cap \mathcal{P}_n(y_n) \in \mathcal{N}(y_n)$. Then

$$\begin{aligned} d_{\mathcal{F}}(\xi, \omega(y_0, \dots, y_n)) &< d_{\mathcal{F}}(\xi', \omega'(y_0, \dots, y_n)) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* \quad \text{by Lemma 15.2} \\ &< d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 \quad \text{by (157)} \\ &< r(x_0, \dots, x_n) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \quad \text{by (178)} \\ &< r(y_0, \dots, y_n) - 3\varepsilon_3 \lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + (2\varepsilon_0 + \varepsilon_3/2) \lambda_{\mathcal{F}}^* \quad \text{by (158)} \\ &< r(y_0, \dots, y_n) + (2\varepsilon_0 - \varepsilon_3) \lambda_{\mathcal{F}}^* < r(y_0, \dots, y_n) \quad \text{by (116.4)} \end{aligned}$$

which contradicts the hypothesis that $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$. Thus, we may assume that

$$(179) \quad r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* < d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) \leq r(x_0, \dots, x_n)$$

Let $1 \leq q \leq p_*$ be the least such q such that there exists $x_n \in \mathcal{X}_{i_n}$ and (179) holds for $\xi' \in \mathcal{X}_q$. Note that $q \neq i_k$ for $0 \leq k \leq n$, and $\mathcal{X}_q = \mathfrak{Z}(\xi_q, \theta_{i_q}, V_0)$ for some $\xi_q \in \Xi$, so that $\xi' = \mathfrak{Z}(\xi_q, \theta_{i_q}, V_0) \cap \mathcal{P}_n(x_n)$.

We now use that ξ_q was chosen inductively to avoid the annular $2\varepsilon_1 \lambda_{\mathcal{F}}^*$ -thickening of the inscribed sphere for each n -simplex in $\Omega^{(n)}(\mathcal{N}(\xi_q))$.

First, we assume that $q > i_n$. For this subcase, we transfer the problem to the plaque $\mathcal{P}_n(\xi_q)$. Set $z_k = \mathcal{X}_{i_k} \cap \mathcal{P}_n(\xi_q)$ for $0 \leq k \leq n$. Note that $\{z_0, \dots, z_n\}$ admits an inscribed sphere by Proposition 18.1, with center $\omega(z_0, \dots, z_n)$ which satisfies

$$(180) \quad d_{\mathcal{F}}(\omega(z_0, \dots, z_n), \omega'(y_0, \dots, y_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

where $\omega'(y_0, \dots, y_n) = \phi_{i_n}(y_n, z_n)(\omega(y_0, \dots, y_n))$. Let $r(z_0, \dots, z_n)$ denote the radius of the sphere, which by (158) satisfies

$$(181) \quad r(y_0, \dots, y_n) - \varepsilon_3 \lambda_{\mathcal{F}}^* \leq r(z_0, \dots, z_n) \leq r(y_0, \dots, y_n) + \varepsilon_3 \lambda_{\mathcal{F}}^*$$

We claim that $\Delta(z_0, \dots, z_n) \in \Delta'_{\mathcal{F}}(\hat{\mathcal{X}}_{i_{q-1}})$. If not, then there exists $\eta' \in \hat{\mathcal{X}}_{i_{q-1}} \cap \mathcal{P}_n(z_n)$ with $d_{\mathcal{F}}(\eta', \omega(z_0, \dots, z_n)) \leq r(z_0, \dots, z_n)$. This contradicts the minimality of the choice of q above. Thus, as ξ_q was chosen to satisfy the inequality (149), we have the estimate

$$(182) \quad d_{\mathcal{F}}(\xi_q, \omega(z_0, \dots, z_n)) > r(z_0, \dots, z_n) + 2\varepsilon_1 \lambda_{\mathcal{F}}^*$$

On the other hand, x_n was chosen so that for the inscribed sphere with center $\omega(x_0, \dots, x_n)$ and radius $r(x_0, \dots, x_n)$ we have the inequality (179) above.

Apply Proposition 18.1 to the cases $x_n \in \mathcal{X}_{i_n}$ and also $z_n \in \mathcal{X}_{i_n}$ to obtain the estimates (158) for both. Together, they imply

$$(183) \quad r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \leq r(z_0, \dots, z_n) \leq r(x_0, \dots, x_n) + 2\varepsilon_3 \lambda_{\mathcal{F}}^*$$

Also, (179) and (183) imply

$$(184) \quad d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \leq r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \leq r(z_0, \dots, z_n)$$

which for $\omega'(x_0, \dots, x_n) = \phi_{i_n}(x_n, z_n)(\omega(x_0, \dots, x_n))$ yields

$$(185) \quad d_{\mathcal{F}}(\xi_q, \omega'(x_0, \dots, x_n)) \leq r(z_0, \dots, z_n) + 2\varepsilon_3 \lambda_{\mathcal{F}}^* + \varepsilon_0 \lambda_{\mathcal{F}}^*$$

and thus by (180) and its corresponding version for x_n yields

$$\begin{aligned} d_{\mathcal{F}}(\xi_q, \omega(z_0, \dots, z_n)) &\leq d_{\mathcal{F}}(\xi_q, \omega'(x_0, \dots, x_n)) + d_{\mathcal{F}}(\omega'(x_0, \dots, x_n), \omega(z_0, \dots, z_n)) \\ &\leq (r(z_0, \dots, z_n) + (2\varepsilon_3 \lambda_{\mathcal{F}}^* + \varepsilon_0 \lambda_{\mathcal{F}}^*)) + (\varepsilon_3 \lambda_{\mathcal{F}}^* + 2\varepsilon_0 \lambda_{\mathcal{F}}^*) \\ (186) \quad &\leq r(z_0, \dots, z_n) + 4\varepsilon_3 \lambda_{\mathcal{F}}^* \end{aligned}$$

which by the choice $\varepsilon_3 < \varepsilon_1/2$ in (111) contradicts (182). Thus, the case $q > i_n$ is not possible.

Finally, consider the case where $q < i_n$. That is, the smallest q such that there exists $x_n \in \mathcal{X}_{i_n}$ and (179) holds for some $\xi' \in \mathcal{X}_q$ occurs for $q < i_n$. This means that in the process of constructing \mathcal{X} , we have chosen a point ξ_q which has distance greater than $2\varepsilon_1 \lambda_{\mathcal{F}}^*$ from all previously inscribed spheres for the net \mathcal{N}_{q-1} , but when we add the point ξ_{i_n} the Delaunay triangulation $\Delta'_{\mathcal{F}}(\mathcal{N}_{i_n})$ abruptly changes on some leaves. The translates of ξ_{i_n} are contained both inside and outside of inscribed spheres, as the translates of ξ_q also wander inside and outside. We show this is impossible, due to

the choice of ε_1 and of the constants ε_3 and ε_4 which control how much the centers of inscribed spheres “wander” for transverse variation at most r_* .

Recall, we assume there is given $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$ and $x_n \in \mathcal{X}_{i_n}$ such that the transverse translate $\{x_0, \dots, x_n\}$ of the set $\{y_0, \dots, y_n\}$ is $\tilde{\rho}_n > 3\varepsilon_2/2$ robust. Thus by Proposition 18.1, there is an inscribed sphere with center $\omega(x_0, \dots, x_n)$ and radius $r(x_0, \dots, x_n)$ which satisfy

$$(187) \quad d_{\mathcal{F}}(\omega(x_0, \dots, x_n), \omega'(y_0, \dots, y_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

$$(188) \quad |r(x_0, \dots, x_n) - r(y_0, \dots, y_n)| < \varepsilon_3 \lambda_{\mathcal{F}}^*$$

where $\omega'(y_0, \dots, y_n) = \phi_{i_n}(y_n, x_n)(\omega(y_0, \dots, y_n))$ is the translate for the center of the inscribed sphere for the n -simplex $\Delta(y_0, \dots, y_n) \in \Delta'_{\mathcal{F}}(\mathcal{X})$. There is also given $1 \leq q < i_n$ so that the translate $\xi' = \mathcal{X}_q \cap \mathcal{P}_n(x_n) \in \mathcal{N}(x_n)$ satisfies (179).

$$(189) \quad r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* < d_{\mathcal{F}}(\xi', \omega(x_0, \dots, x_n)) \leq r(x_0, \dots, x_n)$$

Let $\{x'_0, \dots, x'_n\} = \{x_0, \dots, x_{n-1}, \xi'\}$ denote a reordering of the set so that $x'_k = \mathcal{X}_{i'_k} \cap \mathcal{P}_n(x_n)$ for $0 \leq k \leq n$ with $1 \leq i'_0 < \dots < i'_n \leq p_*$. Then these points satisfy, for $0 \leq k \leq n$,

$$(190) \quad r(x_0, \dots, x_n) - 2\varepsilon_3 \lambda_{\mathcal{F}}^* \leq d_{\mathcal{F}}(x'_k, \omega(x_0, \dots, x_n)) \leq r(x_0, \dots, x_n)$$

The proof of Proposition 17.6 applied to the set $\{x'_0, \dots, x'_n\}$ yields that the collection is $\tilde{\rho}_n$ -robust, so admits an inscribed sphere by Proposition 18.1, with center $\omega(x'_0, \dots, x'_n)$ and radius $r(x'_0, \dots, x'_n)$. From the proof of Proposition 18.1, we have the estimates

$$(191) \quad d_{\mathcal{F}}(\omega(x'_0, \dots, x'_n), \omega(x_0, \dots, x_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

$$(192) \quad |r(x'_0, \dots, x'_n) - r(x_0, \dots, x_n)| < \varepsilon_3 \lambda_{\mathcal{F}}^*$$

Thus, combining (191) and (192), for $\zeta' = x_n$, we obtain

$$(193) \quad d_{\mathcal{F}}(\zeta', \omega(x'_0, \dots, x'_n)) \leq r(x'_0, \dots, x'_n) + 3\varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

Now let $\zeta = \xi_{i_n} \in \mathcal{X}_{i_n}$. The last step is to translate the points $\{x'_0, \dots, x'_n\}$ to the plaque $\mathcal{P}_n(\zeta)$, to obtain points $z'_k = \mathcal{X}_{i'_k} \cap \mathcal{P}_n(\zeta)$. Then $\{z'_0, \dots, z'_n\}$ is $\tilde{\rho}'_n$ -robust by Proposition 17.6, and admits an inscribed sphere with center $\omega(z'_0, \dots, z'_n)$ and radius $r(z'_0, \dots, z'_n)$ by Proposition 18.1. Moreover, this center and radius satisfy

$$(194) \quad d_{\mathcal{F}}(\omega(z'_0, \dots, z'_n), \omega'(x'_0, \dots, x'_n)) \leq \varepsilon_3 \lambda_{\mathcal{F}}^*/2$$

$$(195) \quad |r(z'_0, \dots, z'_n) - r(x'_0, \dots, x'_n)| < \varepsilon_3 \lambda_{\mathcal{F}}^*$$

Combining (193), (194) and (195), we obtain

$$(196) \quad \begin{aligned} d_{\mathcal{F}}(\zeta, \omega(z'_0, \dots, z'_n)) &\leq d_{\mathcal{F}}(\zeta, \omega'(x'_0, \dots, x'_n)) + d_{\mathcal{F}}(\omega'(x'_0, \dots, x'_n), \omega(z'_0, \dots, z'_n)) \\ &\leq d_{\mathcal{F}}(\zeta', \omega(x'_0, \dots, x'_n)) + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 \\ &\leq r(x'_0, \dots, x'_n) + 3\varepsilon_3 \lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 \\ &\leq r(z'_0, \dots, z'_n) + \varepsilon_3 \lambda_{\mathcal{F}}^* + 3\varepsilon_3 \lambda_{\mathcal{F}}^*/2 + 2\varepsilon_0 \lambda_{\mathcal{F}}^* + \varepsilon_3 \lambda_{\mathcal{F}}^*/2 \\ &< r(z'_0, \dots, z'_n) + 4\varepsilon_3 \lambda_{\mathcal{F}}^* + 2\varepsilon_0 \lambda_{\mathcal{F}}^* \end{aligned}$$

By the choice of ε_3 in (111) we have $4\varepsilon_3 + 2\varepsilon_0 < 2\varepsilon_1$, so that (196) contradicts the choice of $\zeta = \xi_{i_n}$ to satisfy

$$d_{\mathcal{F}}(\zeta, \omega(z'_0, \dots, z'_n)) \geq r(z'_0, \dots, z'_n) + 2\varepsilon_1 \lambda_{\mathcal{F}}^*$$

Thus, the case $q < i_n$ again leads to a contraction. This completes the proof of Proposition 18.4. \square

REMARK 18.5. *The sequence of results above shows that \mathcal{X} is a nice stable transversal.*

19. PROOFS OF MAIN THEOREMS

In this last section, we apply the results of the previous sections to prove our main theorems.

19.1. Proof of Theorem 1.1. Let \mathfrak{M} be an equicontinuous matchbox manifold. By Theorem 4.7, the dynamics of its holonomy pseudogroup is minimal. By Theorem 4.8, given any $w_0 \in \mathfrak{T}_*$, say with $w_0 \in \mathfrak{T}_{i_0}$, there exists a descending chain of clopen subsets $\cdots \subset V_{\ell+1} \subset V_\ell \subset \cdots \subset V_0 \subset \mathfrak{T}_{i_0}$ such that for all $\ell \geq 0$, $w_0 \in V_\ell$ and $\text{diam}_{\mathfrak{X}}(V_\ell) < \delta_{\mathcal{U}}^T/2^\ell$. Here, $\delta_{\mathcal{U}}^T$ is the constant of equicontinuity defined in Proposition 4.4, so that each set V_ℓ is in the domain of the holonomy of any path starting at x_0 where $x_0 = \tau_{i_0}(w_0)$. Thus, we can form the associated Thomas tube $\tilde{\mathfrak{N}}(V_\ell)$ as defined by (20).

More is true. Choose w_0 corresponding to a leaf L_0 without holonomy in \mathfrak{M} , let $\tilde{L}_0 \rightarrow L_0$ be the holonomy cover, which is a diffeomorphism, and lift w_0 to a basepoint $\tilde{w}_0 \in \tilde{L}_0$. Then let $\tilde{\mathcal{M}}_0$ be the net in \tilde{L}_0 defined in Section 5.2, with $\tilde{w}_0 \in \tilde{\mathcal{M}}_0$. For $\tilde{z} \in \tilde{\mathcal{M}}_0$ there is a holonomy transport map $h_{\tilde{z}}$ defined by choosing a path from \tilde{w}_0 to \tilde{z} , and we define the holonomy transport of V_ℓ by $V_{\tilde{z}}^\ell = h_{\tilde{z}}(V_\ell)$ as defined by (18). Then the collection of clopen sets $\{V_{\tilde{z}}^\ell \mid \tilde{z} \in \tilde{\mathcal{M}}_0\}$, which covers the transverse space \mathfrak{T}_* , is actually a finite set. Thus, for each ℓ there exists a compact connected subset $\tilde{K}_\ell \subset \tilde{L}_0$ so that V_ℓ is \tilde{K} -admissible, and the Reeb neighborhood, $\tilde{\mathfrak{N}}(\tilde{K}_\ell, V_{\tilde{z}}^\ell)$ as defined by (21), maps onto \mathfrak{M} .

Let $r_* > 0$ be defined by (91)). Apply Theorem 4.4 for $\epsilon = r_*$, to conclude that for ℓ_0 sufficiently large, all holonomy translates $V_{\tilde{z}}^{\ell_0}$ of the set V_{ℓ_0} have diameter less than r_* . We can then apply the methods of Sections 16 and 17 to construct a complete regular V_{ℓ_0} -transversal for \tilde{K}_{ℓ_0} . Then by Theorem 10.1, there exists a foliated homeomorphism into, $\Phi: \tilde{K}_{\ell_0} \times V_{\ell_0} \rightarrow \tilde{\mathfrak{N}}(V_{\ell_0})$, whose image contains the Reeb neighborhood $\tilde{\mathfrak{N}}(\tilde{K}_{\ell_0}, V_{\tilde{z}}^{\ell_0})$. This defines the the transverse Cantor foliation $\tilde{\mathcal{H}}_{\ell_0}$ on a neighborhood of $\tilde{\mathfrak{N}}(\tilde{K}_{\ell_0}, V_{\tilde{z}}^{\ell_0})$ by the methods of Section 10. The transversal $\hat{\mathcal{X}}_{p_*} \subset \tilde{\mathfrak{N}}(V_{\ell_0})$ is invariant, in the sense of Definition 9.3, and thus is the pull-back of a transversal in $\mathfrak{N}(V_{\ell_0}) \subset \mathfrak{M}$. Thus, the transverse Cantor foliation $\tilde{\mathcal{H}}_{\ell_0}$ is Π -equivariant, and descends to a transverse Cantor foliation \mathcal{H}_{ℓ_0} on \mathfrak{M} .

Note that the existence of \mathcal{H}_{ℓ_0} on \mathfrak{M} is the result cited in [20, Theorem 8.3]. In order to complete the proof of Theorem 1.1, note also that [20, Proposition 8.4] shows that the quotient space $M \equiv \mathfrak{M}/\mathcal{H}$ is an n -dimensional topological manifold, and [20, Proposition 8.8] implies that the projection to the leaf space $\mathfrak{M} \rightarrow \mathfrak{M}/\mathcal{H} \cong M$ is a Cantor bundle.

19.2. Proof of Theorem 1.3. Let \mathfrak{M} be a matchbox manifold, $L_x \subset M$ the leaf through $x \in \mathfrak{M}$, and \tilde{L}_x the holonomy covering of L_x . We are given a proper base $K_x \subset L_x$ so that there is a connected compact subset $\tilde{K}_x \subset \tilde{L}_x$ such that the composition $\iota_x: \tilde{K} \subset \tilde{L}_x \rightarrow L_x \subset \mathfrak{M}$ is injective with image K_x .

Introduce the set \hat{K}_x defined by (47) with diameter \hat{R}_K .

Let $w_x = \pi_{i_x}(x)$ be the image of x in a transversal space \mathfrak{T}_{i_x} .

Let $r_* > 0$ be defined by (91). Apply Proposition 4.9 for $\epsilon = r_*/2$, to conclude that there exists $\delta_* = \delta(r_*/2, \hat{R}_K)$, such that if $x \in V_x \subset \mathfrak{T}_{i_x}$ is a clopen neighborhood with $V_x \subset B_{\mathfrak{X}}(w_x, \delta_*)$ then for any path with initial point x and length at most \hat{R}_K the holonomy translate $h_\gamma(V_x)$ of the set V_x has diameter less than r_* . Thus, V_x is \hat{K}_x -admissible, in the sense of Definition 5.8.

Since K_x is compact and the map $\Pi: K_x \rightarrow \mathfrak{M}$ is injective by assumption, by restricting the diameter of the clopen neighborhood V_x further, we may assume that V_x is K_x -disjoint, in the sense of Definition 5.8. In particular, the map $\Pi: \mathfrak{N}(K_x, V_x) \rightarrow \mathfrak{M}$ is injective.

Now apply the methods of Section 10. Choose a basepoint $w_0 \in V_x$ such that the leaf L_0 it defines is without holonomy. Introduce the translated set $\hat{K}_0 \subset \tilde{L}_0$. Then by construction, each translate $h_\gamma(V_x)$ has diameter less than r_* so we can apply the methods of Sections 16 and 17 to

construct a complete regular V_x -transversal for \tilde{K}_0 . Then by Theorem 10.1, there exists a foliated homeomorphism into, $\Phi: \tilde{K}_0 \times V_x \rightarrow \mathfrak{M}$, whose image contains the Reeb neighborhood $\mathfrak{N}(K_x, V_x)$.

There is now a subtle nuance, in that the map Φ no longer need be injective. However, again using that $\Pi: K_x \rightarrow \mathfrak{M}$ is injective, we can chose a clopen sub-neighborhood $x \in V'_x \subset V_x$ such that the restriction $\Phi: K_x \times V'_x \rightarrow \mathfrak{M}$ is injective. This defines a transverse Cantor foliation \mathcal{H} on a neighborhood of $\mathfrak{N}(K_x, V_x)$. This completes the proof of Theorem 1.3.

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